Limites adiabatiques, fibrations holomorphes plates et théorème de R.R.G.

Yeping Zhang

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Yeping ZHANG

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Après avis des rapporteurs :
Jochen BRÜNING (Humboldt-Universität zu Berlin)
Xiaonan MA (Université Paris-Diderot)

Jury de soutenance :

Jean-Michel BISMUT (Université Paris-Sud)	Directeur de thèse
Jean-Benoît BOST (Université Paris-Sud)	Président
Jochen BRÜNING (Humboldt-Universität zu Berlin)	Rapporteur
Gérard FREIXAS I MONTPLET (C.N.R.S.-I.M.J.)	Examinateur
Xiaonan MA (Université Paris-Diderot)	Rapporteur
Werner MÜLLER (Mathematisches Institut der Universität Bonn)	Examinateur
Ken-ichi YOSHIKAWA (Kyoto University)	Examinateur

Thèse préparée au
Département de Mathématiques d'Orsay
Laboratoire de Mathématiques (UMR 8628), Bât. 425
Université Paris-Sud
91405 Orsay CEDEX

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Résumé

Cette thèse est faite de deux parties. La première partie est un article rédigé conjointement avec Martin Puchol et Jialin Zhu. La deuxième partie est une série de résultats obtenus par moi-même liés au théorème de Riemann-Roch-Grothendieck pour les fibrés vectoriels plats.

Nous spécifions les contenus des deux parties. Dans la première partie, nous donnons une preuve analytique d'un résultat décrivant le comportement de la torsion analytique en théorie de de Rham lorsque la variété considérée est séparée en deux par une hypersurface. Plus précisément, nous donnons une formule liant la torsion analytique de la variété entière aux torsions analytiques associées aux variétés à bord avec des conditions limites relative ou absolue le long de l'hypersurface. Ce résultat peut être vu comme une conséquence du théorème de Cheeger-Müller liant la torsion analytique et la torsion combinatoire. Toutefois, le but de notre résultat est d'en avoir une preuve directe en introduisant un cylindre transverse à l'hypersurface dont la longueur tend vers l'infini. La matrice de diffusion introduite par Müller dans ce contexte géométrique joue un rôle important dans la preuve du résultat final.

Dans la deuxième partie de cette thèse, nous raffinons les résultats de Bismut-Lott pour les images directes des fibrés vectoriels plats au cas où le fibré vectoriel plat en question est lui-même la cohomologie holomorphe d'un fibré vectoriel le long d'une fibration plate à fibres complexes. Dans ce contexte, nous donnons une formule de Riemann-Roch-Grothendieck dans laquelle la classe de Todd du fibré tangent relatif apparaît explicitement. En remplaçant les classes de cohomologie par des formes explicites qui les représentent en théorie de Chern-Weil, nous généralisons ainsi des constructions de Bismut-Lott. Plus précisément, si X est une variété réelle compacte, et si $p : \mathcal{N} \to X$ est une fibration plate sur X dont le fibre N est une variété complexes compacte, nous discutions des propriétés du bicomplexe de différentielle $d_X + \overline{\partial}_N$, et nous construisons les formes de torsion analytique associées. Nous démontrons également des propriétés fonctorielles de ces formes.

Abstract

This thesis consists of two parts. The first part is an article written jointly with Martin Puchol and Jialin Zhu, the second part is a series of results obtained by myself in connection with the Riemann-Roch-Grothendieck theorem for flat vector bundles.

Let us be more specific on the content of these two parts. In the first part, we give an analytic approach to the behavior of classical Ray-Singer analytic torsion in de Rham theory when a manifold is separated along a hypersurface. More precisely, we give a formula relating the analytic torsion of the full manifold, and the analytic torsion associated with relative or absolute boundary conditions along the hypersurface. This result can also be viewed as a consequence of the Cheeger-Müller theorem that relates analytic torsion to combinatorial torsion. However, the point of our proof is to obtain a direct proof of this result, by introducing a cylinder transversal to the hypersurface whose length is made to tend to $+\infty$. The scattering matrix introduced by Müller in this geometric context plays an important role in establishing the final result.

In the second part of this thesis, we refine the results of Bismut-Lott on direct images of flat vector bundles to the case where the considered flat vector bundle is itself the fiberwise holomorphic cohomology of a vector bundle along a flat fibration by complex manifolds. In this context, we give a formula of Riemann-Roch-Grothendieck in which the Todd class of the relative holomorphic tangent bundle appears explicitly. By replacing cohomology classes by explicit differential forms in Chern-Weil theory, we extend the constructions of Bismut-Lott in this context. More precisely, if X is a compact real manifold, and if $p : \mathcal{N} \to X$ is a flat fibration over X whose fiber N is a compact complex manifold, we discuss the properties of the bicomplex with chain map $d_X + \overline{\partial}_N$. In this context, we construct explicit analytic torsion forms which transgress the equality of cohomology classes at the level of differential forms, and we establish corresponding functorial properties of these new analytic torsion forms.

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Contents

1. Scattering matrix and analytic torsion

1.0. **Introduction.**

By a non-compact Riemannian manifold with cylindrical ends, we mean a Riemannian manifold having an open subset isometric to an infinite cylinder such that the complement is compact. Such a manifold could be associated with a scattering matrix, which encodes how an incoming wave on the cylinder is scattered by the compact part. In [M94], Müller studied the η-invariants of non-compact Riemannian manifolds with cylindrical ends using the scattering matrix.

Now we consider a compact Riemannian manifold containing an open subset isometric to a finite cylinder. We deform the metric in such a way that the length of the cylinder tends to infinity. This process is referred to as taking the adiabatic limit (see [BC89, BF86] for another kind of adiabatic limit). It first appeared in Douglas-Wojciechowski's work [DW91] on η-invariants. Cappell-Lee-Miller [CLM96] studied the asymptotic behavior of Laplacians under the adiabatic limit. They showed that an eigenvalue of the Laplacian either tends to zero (small eigenvalue) or remains uniformly away from zero (large eigenvalue). Park-Wojciechowski [PW06] showed that the asymptotic behavior of certain small eigenvalues is determined by the scattering matrices obtained as follows : the manifold in question converges to the disjoint union of two non-compact Riemannian manifolds with cylindrical ends, each of which gives us a scattering matrix.

In this paper, we concentrate on the asymptotic behavior of Hodge-de Rham operators, a special kind of Dirac operator, under the adiabatic limit. The scattering matrix plays a key role in our research.

One of our main results is an asymptotic estimate of the spectrum of Hodge-de Rham operator under the adiabatic limit. As a consequence, we get an asymptotic gluing formula for the ζ-determinant of the Hodge-Laplacian (square of the Hodge-de Rham operator).

Another main result is an asymptotic estimate of the L^2-metric on the de Rham cohomology group in the adiabatic limit. As a consequence, we get the adiabatic limit of the torsion associated with the Mayer-Vietoris exact sequence.

Applying the results mentioned above, we obtain an analytic proof of the gluing formula for analytic torsion.

Let us explain the analytic torsion in more detail. For a flat complex vector bundle F equipped with a Hermitian metric over a compact Riemannian manifold Z, its Ray-Singer analytic torsion [RS71] is a (weighted) product of the determinants of the Hodge-Laplacian twisted by F. The Ray-Singer metric on $\det H^\bullet(Z,F)$ is the product of its L^2-metric and the Ray-Singer analytic torsion. The Ray-Singer metric has a topological counterpart, known as the Reidemeister metric [Rei35]. Ray and Singer [RS71] conjectured that the two metrics coincide. For unitarily flat vector bundles, this conjecture was proved independently by Cheeger [Che79] and Müller [M78]. Bismut-Zhang [BZ92] and Müller [M93] simultaneously considered generalizations of this result. Müller [M93] extended this result to the case where the dimension of the manifold is odd and only the metric induced on $\det F$ is required to be flat. Bismut-Zhang [BZ92] generalized this result to arbitrary flat vector bundles with arbitrary Hermitian metrics. There are also various extensions to the equivariant case [LoR91, Lüc93, BZ94].

Assume that there is a hypersurface $Y \subseteq Z$ cutting Z into two submanifolds $Z_1, Z_2 \subseteq Z$, it is natural to expect an additive formula linking the analytic torsions associated with Z_1, Z_2 and Z. This problem was first formulated by Ray-Singer [RS71] as a possible approach to Ray-Singer conjecture. It was proved for unitarily flat vector bundles with product structure metrics near Y by Lück [Lüc93], Vishik [V95], and proved in full generality by Brüning-Ma [BM13]. There are also related works of [H98] and [L13].

The family version of the analytic torsion was constructed by Bismut-Lott [BL95] (BL-torsion). Under the hypothesis that there exists a fiberwise Morse function, Bismut-Goette [BGo1] obtained a family version of the Bismut-Zhang theorem, i.e., a formula linking BL-torsion to higher Reidemeister torsion ([I02, DWW03, BDKW11], see also [Goe09] for a survey). It is conjectured (conference on the higher torsion invariants, Göttingen, September 2003) that there should exist a gluing formula for BL-torsion. This conjecture may serve as an intermediate step in establishing the relation between the BL-torsion and the higher Reidemeister torsion in full generality, conjectured by Igusa [I08]. Zhu [Zhu15] established the desired formula under the same hypothesis as Bismut-Goette's [BGo1].

Our proof of the gluing formula is analytic. It could be generalized for BL-torsion. Our strategy was applied by Zhu [Zhu] to prove the gluing formula for BL-torsion under the hypothesis $H^\bullet(Y,F)=0$. We remark that $H^\bullet(Y,F)=0$ implies the absence of s-values (cf. §1.0.2) and the splitting of the Mayer-Vietoris exact sequence.

Let us now give more detail about the matter of this paper.

1.0.1. *Manifolds with cylindrical ends and scattering matrices.*

Let X be a compact manifold with boundary $\partial X = Y$. We fix $U =]-1,0]\times Y$ a collar neighborhood of ∂X. Let $\pi_Y :]-1,0]\times Y \to Y$ be the natural projection. Let F be a flat complex vector bundle over X with flat connection ∇^F. Using parallel transport along $u\in]-1,0]$, $(F|_U,\nabla^F|_U)$ is identified with $\pi_Y^*(F|_Y,\nabla^F|_U)$ (cf. (1.2.7)).

We equip X with a Riemannian metric g^{TX} and F with a Hermitian metric h^F. Let g^{TY} be the metric on Y induced by g^{TX}. We suppose that (cf. [BM13, (2.1) and (2.3)])

$$g^{TX}\big|_U = du^2 + g^{TY}\ , \quad h^F\big|_U = \pi_Y^*\left(h^F\big|_Y\right)\ . \tag{1.0.1}$$

For $0\leqslant R\leqslant\infty$, set $X_R = X\cup_Y [0,R]\times Y$. We call $U_R := U\cup[0,R]\times Y =]-1,R]\times Y$ the cylindrical part of X_R. Let $\pi_Y :]-1,R]\times Y\to Y$ be the natural projection. Then F extends to X_R in the natural way : $(F,\nabla^F)\big|_{U_R} = \pi_Y^*\left(F\big|_Y,\nabla^F\big|_Y\right)$. We extend equally g^{TX} and h^F to X_R in such a way that (1.0.1) holds with U replaced by U_R.

Let $\Omega^\bullet(X_R,F)$ be the vector space of differential forms on X_R with values in F. Let $d^F:\Omega^\bullet(X_R,F)\to\Omega^{\bullet+1}(X_R,F)$ be the de Rham operator induced by ∇^F, let $d^{F,*}$ be its formal adjoint (with respect to L^2-metric). The Hodge-de Rham operator is defined by

$$D^F_{X_R} = d^F + d^{F,*}\ . \tag{1.0.2}$$

Its square $D^{F,2}_{X_R}$ is the Hodge-Laplacian.

For $R=\infty$, the spectrum of $D^{F,2}_{X_\infty}$ has an absolutely continuous part (cf. [RS80, §7.2]).

Let $\mathscr{H}^\bullet(Y,F)\subseteq\Omega^\bullet(Y,F)$ be the kernel of D^F_Y, the Hodge-de Rham operator on $\Omega^\bullet(Y,F)$. Set $\mathscr{H}^\bullet(Y,F[du]) = \mathscr{H}^\bullet(Y,F)\oplus\mathscr{H}^\bullet(Y,F)du$. We fix $\delta_Y>0$ such that $]-\delta_Y,\delta_Y[\cap\mathrm{Sp}(D^F_Y)\subseteq\{0\}$. The scattering matrix (cf. [K65, Theorem 1], [M94, §4])

$$C(\lambda)\in\mathrm{End}\big(\mathscr{H}^\bullet(Y,F[du])\big)\ ,\quad \lambda\in]-\delta_Y,\delta_Y[\ , \tag{1.0.3}$$

is characterized by the following property : for ω a generalized eigensection (cf. §1.2.3) of $D^F_{X_\infty}$ with eigenvalue $\lambda\in]-\delta_Y,\delta_Y[$, there exist $\phi\in\mathscr{H}^\bullet(Y,F[du])$ and

$$\theta\in\mathscr{C}^\infty\big([0,\infty[\,,\Omega^\bullet(Y,F[du])\big)\ , \tag{1.0.4}$$

which is L^2-integrable, such that (cf. (1.2.31))

$$\omega\big|_{U_\infty} = e^{-i\lambda u}\phi + e^{i\lambda u}C(\lambda)\phi + \theta \,. \tag{1.0.5}$$

1.0.2. *Asymptotics of the spectrum of Hodge-Laplacian.*

Let (Z, g^{TZ}) be a closed Riemannian manifold. Let $Y \subseteq Z$ be a hypersurface cutting Z into two pieces, say Z_1 and Z_2. Then $\partial Z_1 = \partial Z_2 = Y$ and $Z = Z_1 \cup_Y Z_2$. Let (F, ∇^F) be a flat complex vector bundle over Z. Its restriction to Z_1 or Z_2 is still denoted by F. Let h^F be a Hermitian metric on F. We suppose that g^{TZ} and h^F have product structure near Y, in the sense of (1.0.1).

Proceeding in the same way as §1.0.1, we construct the Riemannian manifold $Z_{j,R}$ $(j = 1, 2)$, which is Z_j with a cylinder of length R attached. For $R \in [0, \infty[$, set $Z_R = Z_{1,R} \cup_Y Z_{2,R}$. Then (F, ∇^F, h^F) extends to Z_R in the sense of (1.0.1) and (1.2.7) .

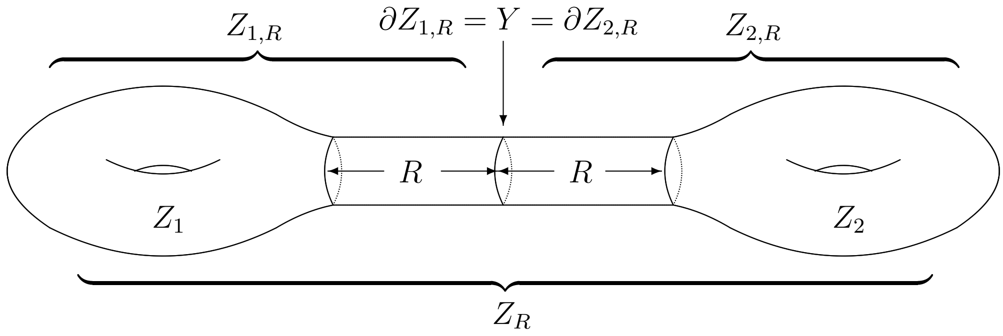

Figure 1

In the whole paper, we will always put the relative boundary condition on $Z_{1,R}$ and put the absolute boundary condition on $Z_{2,R}$ (cf. (1.1.5)**).** Let $D^F_{Z_R}$ be the Hodge-de Rham operator (cf. (1.0.2)) acting on $\Omega^\bullet(Z_R, F)$. We define equally $D^F_{Z_{j,R}}$ $(j = 1, 2)$, the Hodge-de Rham operator acting on $\Omega^\bullet_{\mathbf{bd}}(Z_{j,R}, F)$ (cf. (1.1.5)).

The eigenvalues of $D^F_{Z_R}$ are classified by Cappell-Lee-Miller [CLM96, Theorem A] according to their asymptotic behaviors as $R \to \infty$:

- large eigenvalue (l-value), which remains uniformly away from 0;
- polynomially small eigenvalue (s-value), which tends to zero with speed slower than $R^{-1-\varepsilon}$ for any $\varepsilon > 0$;
- exponentially small eigenvalue (e-value), which lies in $[-e^{-cR}, e^{-cR}]$ for certain $c > 0$.

Moreover, there are only finitely many exponentially small eigenvalues. Park-Wojciechowski [PW06, Theorem 3.5] gave an estimate of the the s-values lying in $[-R^{-\varepsilon}, R^{-\varepsilon}]$ in term of the scattering matrix. They also showed that the e-values are identically zero if $\nabla^F h^F = 0$ [PW06, Proposition 3.9].

In this paper, we show that (see Theorem 1.3.18) : for a Hodge-de Rham operator, there exists $\delta > 0$, such that the estimate (1.3.142) holds for s-values lying in $[-\delta, \delta]$, furthermore, all the e-values are identically zero. We also extend our results to manifolds with boundaries equipped with relative/absolute boundary condition (see Theorem 1.4.7). As a consequence, we get an asymptotic gluing formula for the ζ-determinants under the adiabatic limit, stated in the sequel.

Let N be the number operator on $\Omega^\bullet(Z_R,F)$, i.e., for $\omega\in\Omega^p(Z_R,F)$, $N\omega=p\omega$. Let $P:\Omega^\bullet(Z_R,F)\to\ker\left(D^{F,2}_{Z_R}\right)$ be the orthogonal project with respect to the L^2-metric. The ζ-function associated with $D^{F,2}_{Z_R}$ is defined, for $s\in\left\{\mathbb{C}\ :\ \mathrm{Re}(s)>\frac{1}{2}\dim Z\right\}$, by

$$\zeta_R(s)=-\operatorname{Tr}\left[(-1)^N N\left(D^{F,2}_{Z_R}\right)^{-s}(1-P)\right]\ . \tag{1.0.6}$$

Then ζ_R admits a meromorphic extension to the whole complex plane $\mathbb{C}$, which is regular at $0\in\mathbb{C}$. Let $\zeta^p_R(s)$ be (1.0.6) with $D^{F,2}_{Z_R}$ replaced by $D^{F,2,(p)}_{Z_R}:=D^{F,2}_{Z_R}\big|_{\Omega^p(Z_R,F)}$. Then

$$\exp\left(\zeta_R{}'(0)\right)=\prod_{p=1}^{\dim Z}\Big(\exp\left(\zeta^p_R{}'(0)\right)\Big)^p\ , \tag{1.0.7}$$

i.e., it is a weighted product of the ζ-determinants of $D^{F,2,(p)}_{Z_R}$. We call $\exp\left(\zeta_R{}'(0)\right)$ the (weighted) ζ-determinant of $D^{F,2}_{Z_R}$. In the same way, we define $\zeta_{j,R}(s)$, the ζ-function associated with $D^{F,2}_{Z_{j,R}}$.

Let $C_j(\lambda)\in\operatorname{End}\big(\mathscr{H}^\bullet(Y,F[du])\big)$ ($j=1,2$, $\lambda\in\mathbb{R}$) be the scattering matrix associated with $\Omega^\bullet(Z_{j,\infty},F)$. For $p=0,\cdots,\dim Z$, we denote

$$C_{12}=\left(C_2^{-1}C_1\right)(0)\ ,\qquad C^p_{12}=C_{12}\big|_{\mathscr{H}^p(Y,F)\oplus\mathscr{H}^{p-1}(Y,F)du}\ . \tag{1.0.8}$$

Set

$$\begin{aligned}
\chi'(C_{12})&=\sum_{p=0}^{\dim Z}p(-1)^p\dim\ker\left(C^p_{12}-1\right)\ ,\\
\chi'&=\sum_{p=0}^{\dim Z}p(-1)^p\Big\{\dim H^p(Z,F)\\
&\qquad\qquad-\dim H^p_{\mathbf{bd}}(Z_1,F)-\dim H^p_{\mathbf{bd}}(Z_2,F)\Big\}\ ,\\
\chi(Y,F)&=\sum_{p=0}^{\dim Y}(-1)^p\dim H^p(Y,F)\ ,
\end{aligned} \tag{1.0.9}$$

where $H^\bullet_{\mathbf{bd}}(\cdot,F)$ is defined by (1.0.24).

For a Hermitian matrix A, we denote by $\det{}^*(A)$ be the product of its non zero eigenvalues.

Theorem 1.0.1. *For any $\varepsilon>0$, as $R\to+\infty$, we have*

$$\begin{aligned}
&\zeta_R{}'(0)-\zeta_{1,R}{}'(0)-\zeta_{2,R}{}'(0)\\
&=2\chi'\log R+\big(\chi(Y,F)+\chi'(C_{12})\big)\log 2\\
&\quad+\sum_{p=0}^{\dim Z}\frac{p}{2}(-1)^p\log\det{}^*\Big(\frac{2-C^p_{12}-(C^p_{12})^{-1}}{4}\Big)+\mathscr{O}(R^{-1+\varepsilon})\ .
\end{aligned} \tag{1.0.10}$$

We remark that the asymptotic gluing formulas for the ζ-determinants in different contexts were studied by Müller-Müller [MM06] and Park-Wojciechowski [PW06].

1.0.3. *Analytic torsion and Mayer-Vietoris exact sequence.*

For a complex line λ, let $\lambda^{-1} = \lambda^*$ be its dual. For E a finite dimensional complex vector space, its determinant line is defined by $\det E = \Lambda^{\max} E$. More generally, for a $\mathbb{Z}$-graded finite dimensional vector space $E^\bullet = \bigoplus_{k=0}^n E^k$, we define

$$\det E^\bullet = \bigotimes_{k=0}^n \left(\det E^k\right)^{(-1)^k} . \tag{1.0.11}$$

For

$$(V^\bullet, \partial) : 0 \to V^0 \to V^1 \to \cdots \to V^n \to 0 \tag{1.0.12}$$

an exact sequence of finite dimensional complex vector spaces, there is a canonical section $\varrho \in \det V^\bullet$: let $m_j = \dim \operatorname{im} \left(\partial\big|_{V^j}\right)$, we choose $(s_{j,k})_{1\leqslant k\leqslant m_j}$ in V^j such that they project to a basis of $V^j/\partial V^{j-1}$, then with $\wedge_k s_{j,k} := s_{j,1}\wedge\cdots\wedge s_{j,m_j}$, we define

$$\varrho = \bigotimes_{j=0}^n \Big((\wedge_k \partial s_{j-1,k}) \wedge (\wedge_k s_{j,k}) \Big)^{(-1)^j} \in \det V^\bullet . \tag{1.0.13}$$

Let $g^{V^\bullet}$ be a Hermitian metric on $V^\bullet$. Let ∂^* be the adjoint of ∂. Then $(\partial+\partial^*)^2 = \partial\partial^* + \partial^*\partial$ preserves each V^j. The torsion (cf. [BGS88a, Definition 1.4]) associated with $(V^\bullet, \partial)$ is defined by

$$\mathscr{T}(V^\bullet, \partial) = \prod_j \left[\det\left((\partial+\partial^*)^2\big|_{V^j}\right)\right]^{(-1)^j j/2} \in \mathbb{R}_+ . \tag{1.0.14}$$

Let $\|\cdot\|_{\det V^\bullet}$ be the metric on $\det V^\bullet$ induced by $g^{V^\bullet}$. We have (cf. [BGS88a, Proposition 1.5])

$$\mathscr{T}(V^\bullet, \partial) = \|\varrho\|_{\det V^\bullet} . \tag{1.0.15}$$

We recall that $Z_{1,R}$, $Z_{2,R}$, Z_R and F are defined in §1.0.2. We consider the following Mayer-Vietoris exact sequence

$$\cdots \to H^p_{\mathbf{bd}}(Z_{1,R}, F) \to H^p(Z_R, F) \to H^p_{\mathbf{bd}}(Z_{2,R}, F) \to \cdots , \tag{1.0.16}$$

which is equipped with L^2-metrics. We denote by $\mathscr{T}_R$ its torsion.

Theorem 1.0.2. *As $R\to\infty$, we have*

$$\mathscr{T}_R = 2^{\chi'(C_{12})/2} R^{\chi'} \prod_{p=0}^{\dim Z} \det{}^*\Big(\frac{2 - C^p_{12} - (C^p_{12})^{-1}}{4}\Big)^{\frac{p}{4}(-1)^p} + \mathscr{O}(R^{\chi'-1}) . \tag{1.0.17}$$

Viewing the Mayer-Vietoris exact sequence (1.0.16) with $R = 0$ as an acyclic complex and applying (1.0.13), we get the canonical section

$$\varrho \in \lambda(F) := \Big(\det H^\bullet(Z,F)\Big)^{-1} \otimes \det H^\bullet_{\mathbf{bd}}(Z_1, F) \otimes \det H^\bullet_{\mathbf{bd}}(Z_2, F) . \tag{1.0.18}$$

We use the conventions that $Z_0 = Z$ and $H^\bullet_{\mathbf{bd}}(Z_0, F) = H^\bullet(Z,F)$. Let $\zeta_j(s)$ ($j = 0,1,2$) be the ζ-functions (cf. (1.0.6)) associated with the Hodge-Laplacian $D^{F,2}_{Z_j}$. Let $\|\cdot\|^{L^2}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)}$ be the L^2-metric on $\det H^\bullet_{\mathbf{bd}}(Z_j, F)$.

The Ray-Singer metric on $\det H^\bullet_{\mathbf{bd}}(Z_j,F)$ ($j = 0,1,2$) is defined by

$$\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)} = \|\cdot\|^{L^2}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)} \exp\left(\frac{1}{2}\zeta_j{}'(0)\right) . \tag{1.0.19}$$

Let $\|\cdot\|^{\mathrm{RS}}_{\lambda(F)}$ be the product norm on $\lambda(F)$ induced by $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)}$. The following theorem is first proved by Brüning-Ma [BM13, Theorem 0.3].

Theorem 1.0.3. *If g^{TZ} and h^F have product structures near Y (cf. (1.0.1)), then*

$$\|\varrho\|^{\mathrm{RS}}_{\lambda(F)} = 2^{-\frac{1}{2}\chi(Y,F)} \; . \tag{1.0.20}$$

Let $\mathscr{T} = \mathscr{T}_0$. Then (1.0.20) can be reformulated as follows.

$$\frac{1}{2}\zeta'(0) - \frac{1}{2}\zeta_1{}'(0) - \frac{1}{2}\zeta_2{}'(0) - \log \mathscr{T} = \frac{1}{2}\chi(Y,F)\log 2 \; . \tag{1.0.21}$$

In this paper, we give a direct proof of (1.0.21) : by Theorem 1.0.1, 1.0.2, we know that $t_R := \frac{1}{2}\zeta_R'(0) - \frac{1}{2}\zeta_{1,R}'(0) - \frac{1}{2}\zeta_{2,R}'(0) - \log \mathscr{T}_R$ tends to $\frac{1}{2}\chi(Y,F)\log 2$ as $R\to\infty$, meanwhile, using the anomaly formula for the analytic torsion [BZ92, Theorem 0.1], we know that t_R is independent of R. This proves (1.0.21).

This paper is organized as follows. In §1.1, we review some results concerning the absolute/relative cohomology of manifolds with boundaries and the Mayer-Vietoris exact sequence. In §1.2, we review some results concerning the spectrum of the Hodge-Laplacian on a manifold with cylindrical ends and introduce the scattering matrix. In §1.3, we study the spectrum of the Hodge-Laplacian on the stretched manifold Z_R, and link it to the scattering theory. In §1.4, we prove similar results for manifolds with boundary. In §1.5, we prove Theorem 1.0.1. In §1.6, we prove Theorem 1.0.2. In §1.7, we give our new proof of Theorem 1.0.3.

1.0.4. *Notations.*

Hereby, we summarize some frequently used notations in this paper.

A manifold (with or without boundary) is usually denoted by X, Y or Z. We denote by g^{TX} a Riemannian metric on X. We always consider a manifold equipped with a flat complex vector bundle F, a flat connection ∇^F and a Hermitian metric h^F.

By $\Omega^\bullet(X,F)$, we mean the vector space of differential forms on X with values in F. We denote by $\Omega^\bullet_{\mathrm{c}}(X,F)$ the subspace of differential forms that are compactly supported.

By $\|\cdot\|_X$, we mean the L^2-metric on $\Omega^\bullet(X,F)$. More precisely, let $\langle\cdot,\cdot\rangle_{\Lambda^\bullet(T^*X)\otimes F}$ be the scalar product on $\Lambda^\bullet(T^*X)\otimes F$ induced by g^{TX} and h^F. Let dv_X be the Riemannian volume form on X, then, for $\omega\in\Omega^\bullet(X,F)$, we have

$$\|\omega\|^2_X = \int_X \langle\omega_x,\omega_x\rangle_{\Lambda^\bullet(T^*X)\otimes F} dv_X(x) \; . \tag{1.0.22}$$

The scalar product associated with $\|\cdot\|_X$ is denoted by $\langle\cdot,\cdot\rangle_X$. By $\|\cdot\|_{\mathscr{C}^0,X}$, we mean the $\mathscr{C}^0$-norm on $\Omega^\bullet(X,F)$. More precisely,

$$\|\omega\|^2_{\mathscr{C}^0,X} = \sup\Big\{\langle\omega_x,\omega_x\rangle_{\Lambda^\bullet(T^*X)\otimes F} \; : \; x\in X\Big\} \; . \tag{1.0.23}$$

By d^F, we mean the de Rham operator acting on $\Omega^\bullet(X,F)$ induced by ∇^F. By $d^{F,*}$, we mean the formal adjoint of d^F. The Hodge-de Rham operator is defined by $D^F_X = d^F + d^{F,*}$.

We denote

$$H^\bullet_{\mathrm{abs}}(X,F) = H^\bullet(X,F) \; , \quad H^\bullet_{\mathrm{rel}}(X,F) = H^\bullet(X,\partial X,F) \; . \tag{1.0.24}$$

We write $H^\bullet_{\mathbf{bd}}(X,F)$ for short if the choice of abs/rel is clear.

By the L^2-metric on $H^\bullet_{\mathbf{bd}}(X,F)$, we mean the metric induced by $\|\cdot\|_X$ via Hodge theorem (cf. Theorem 1.1.1).

If A is a self-adjoint operator, we denote by $\mathrm{Sp}(A)$ its spectrum.

For a Hermitian matrix A, we note

$$\det{}^*(A) = \prod_{\lambda \in \mathrm{Sp}(A)\setminus\{0\}} \lambda \,. \tag{1.0.25}$$

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1.1. **Cohomologies for manifolds with boundary.**

In this section, we review some basic constructions/results concerning the cohomology of a compact manifold with boundary.

In §1.1.1, using the language of simplical complex, we define the absolute/relative cohomology of a compact manifold with boundary with values in a flat vector bundle. In §1.1.2, we state the Hodge theorem for the absolute/relative cohomology. In §1.1.3, we state the classical Mayer-Vietoris exact sequence in the language of the simplicial cohomology together with its interpretation in the language of the de Rham cohomology and the Hodge theory.

1.1.1. *Absolute/Relative cohomology.*

Let X be a compact $\mathscr{C}^\infty$-manifold with boundary $\partial X = Y$. Let $F \to X$ be a flat complex vector bundle equipped with a flat connection ∇^F. Let F^* be the dual vector bundle of F.

Let K_X be a smooth triangulation of X, such that $K_Y = K_X \cap Y$ gives a triangulation of Y. For $0 \leqslant p \leqslant \dim X$, let $K_X^p \subseteq K_X$ be the set of cells in K_X of dimension $\leqslant p$. Let B be the set of barycenters of the simplexes in K_X. Let $b : K_X \to B$ be the obvious one-to-one map. If $\mathfrak{a} \in K_X$, let $[\mathfrak{a}]$ be the real line generated by $\mathfrak{a}$. Let $(C_\bullet(K_X, F^*), \partial)$ be the complex of simplical chains in K_X with values in F^*. We have

$$C_p(K_X, F^*) = \bigoplus_{\mathfrak{a} \in K_X^p \setminus K_X^{p-1}} [\mathfrak{a}] \otimes_{\mathbb{R}} F^*_{b(\mathfrak{a})} \,. \tag{1.1.1}$$

The chain map ∂ maps $C_p(K_X, F^*)$ to $C_{p-1}(K_X, F^*)$. Then $(C_\bullet(K_Y, F^*), \partial)$ is a subcomplex of $(C_\bullet(K_X, F^*), \partial)$. We define the quotient complex

$$C_\bullet(K_X/K_Y, F^*) = C_\bullet(K_X, F^*)/C_\bullet(K_Y, F^*) \,. \tag{1.1.2}$$

For $\mathfrak{a} \in K_X$, let $[\mathfrak{a}]^*$ be the real line dual to $[\mathfrak{a}]$. Let $(C^\bullet(K_X, F), \widetilde{\partial})$ be the complex dual to $(C_\bullet(K_X, F^*), \partial)$, more precisely,

$$C^p(K_X, F) = \bigoplus_{\mathfrak{a} \in K_X^p \setminus K_X^{p-1}} [\mathfrak{a}]^* \otimes_{\mathbb{R}} F_{b(\mathfrak{a})} \simeq \big(C_p(K_X, F^*)\big)^* \,, \tag{1.1.3}$$

and $\widetilde{\partial}$ is dual to ∂. Let $C^p(K_X/K_Y, F)$ be the maximal subset of $C^p(K_X, F)$, whose pairing with $C_p(K_Y, F^*)$ is zero. Then $(C^\bullet(K_X/K_Y, F), \widetilde{\partial})$ is a sub complex of $(C^\bullet(K_X, F), \widetilde{\partial})$.

We define

$$
\begin{aligned}
H^\bullet(X,F) &= H^\bullet\left(C^\bullet(K_X,F),\widetilde{\partial}\right)\ ,\\
H^\bullet(X,\partial X,F) &= H^\bullet\left(C^\bullet(K_X/K_Y,F),\widetilde{\partial}\right)\ .
\end{aligned}
\tag{1.1.4}
$$

1.1.2. *Hodge Theorem.*

Let g^{TX} be a Riemannian metric on X. Let h^F be a Hermitian metric on F. We identify a neighborhood of ∂X to $]-1,0]\times Y$. Let (u,y) ($u\in]-1,0]$, $y\in Y$) be its coordinates. We suppose that (1.0.1) holds.

We equip ∂X with the absolute/relative boundary condition :

$$
\begin{aligned}
\Omega^\bullet_{\mathrm{abs}}(X,F) &:= \left\{\omega\in\Omega^\bullet(X,F)\ :\ i_{\frac{\partial}{\partial u}}\omega = 0 \text{ on } Y\right\}\ ,\\
\Omega^\bullet_{\mathrm{rel}}(X,F) &:= \left\{\omega\in\Omega^\bullet(X,F)\ :\ du\wedge\omega = 0 \text{ on } Y\right\}\ .
\end{aligned}
\tag{1.1.5}
$$

We write $\Omega^\bullet_{\mathbf{bd}}(X,F)$ for short if the choice of abs/rel is clear.

Let $d^{F,*}$ be the formal adjoint of the de Rham operator d^F with respect to the L^2-metric $\langle\cdot,\cdot\rangle_X$ (cf. §1.0.4). The Hodge-de Rham operator acting on $\Omega^\bullet_{\mathbf{bd}}(X,F)$ is defined by

$$
D^F_X = d^F + d^{F,*}\ .
\tag{1.1.6}
$$

Set

$$
\begin{aligned}
\Omega^\bullet_{\mathrm{abs},D^2}(X,F) &:= \left\{\omega\in\Omega^\bullet(X,F)\ :\ i_{\frac{\partial}{\partial u}}\omega = 0\ ,\ i_{\frac{\partial}{\partial u}}d^F\omega = 0 \text{ on } Y\right\}\ ,\\
\Omega^\bullet_{\mathrm{rel},D^2}(X,F) &:= \left\{\omega\in\Omega^\bullet(X,F)\ :\ du\wedge\omega = 0\ ,\ du\wedge d^{F,*}\omega = 0 \text{ on } Y\right\}\ .
\end{aligned}
\tag{1.1.7}
$$

We write $\Omega^\bullet_{\mathbf{bd},D^2}(X,F)$ for short if the choice of abs/rel is clear.

Let $D^{F,2}_X$ act on $\Omega^\bullet_{\mathbf{bd},D^2}(X,F)$.

Let $\Omega^\bullet_{L^2}(X,F)$ be the completion of $\Omega^\bullet(X,F)$ with respect to $\langle\cdot,\cdot\rangle_X$.

We define the de Rham map $P_\infty:\Omega^\bullet(X,F)\to C^\bullet(K_X,F)$ by

$$
P_\infty(\sigma)([\mathfrak{a}]\otimes v) = \int_{\mathfrak{a}}(\sigma,v)\ ,
\tag{1.1.8}
$$

where $\mathfrak{a}\in K_X$, $v\in F^*_{b(\mathfrak{a})}$, $\sigma\in\Omega^\bullet(Z,F)$.

The following Hodge theorem is proved in [RS71, Prposition 4.2, Corollary 5.7] in the case $\nabla^F h^F = 0$. The fact that the same proof works in the general case is noticed in [BM13, Theorem 1.1].

Theorem 1.1.1. *We have*

$$
\ker\left(D^{F,2}_X\right) = \ker\left(D^F_X\right) = \ker\left(d^F\right)\cap\ker\left(d^{F,*}\right)\cap\Omega^\bullet_{\mathbf{bd}}(X,F)\ .
\tag{1.1.9}
$$

The vector space $\ker\left(D^F_X\right)$ *is finite dimensional.*

The following orthogonal decompositions hold,

$$
\begin{aligned}
\Omega^p_{\mathbf{bd}}(X,F) &= \ker\left(D^F_X\right)\oplus d^F\Omega^{p-1}_{\mathbf{bd},D^2}(X,F)\oplus d^{F,*}\Omega^{p+1}_{\mathbf{bd},D^2}(X,F)\ ,\\
\Omega^p_{L^2}(X,F) &= \ker\left(D^F_X\right)\oplus \overline{d^F\Omega^{p-1}_{\mathbf{bd},D^2}(X,F)}\oplus \overline{d^{F,*}\Omega^{p+1}_{\mathbf{bd},D^2}(X,F)}\ ,
\end{aligned}
\tag{1.1.10}
$$

where $\overline{\ \cdot\ }$ *denotes the* L^2*-closure.*

For the absolute (resp. relative) boundary condition, the inclusion $\ker\left(D_X^F\right) \hookrightarrow \ker\left(d^F\right) \cap \Omega^\bullet_{\mathbf{bd}}(X,F)$ *composed with the de Rham map* P_∞ *maps into the vector space of cocycles in* $C^\bullet(K_X,F)$ *(resp.* $C^\bullet(K_X/K_Y,F)$*). We obtain an isomorphism*

$$P_\infty : \ker\left(D_X^{F,2}\right) \to H^\bullet_{\mathbf{bd}}(X,F)\ . \tag{1.1.11}$$

We define

$$H^p\left(\Omega^\bullet_{\mathbf{bd}}(X,F),d^F\right) = \frac{\ker\left(d^F\right)\cap\Omega^p_{\mathbf{bd}}(X,F)}{d^F\Omega^{p-1}_{\mathbf{bd}}(X,F)\cap\Omega^p_{\mathbf{bd}}(X,F)}\ . \tag{1.1.12}$$

By Theorem 1.1.1, P_∞ induces the following isomorphisms

$$\begin{aligned} H^p\left(\Omega^\bullet_{\mathrm{abs}}(X,F),d^F\right) &\simeq H^p_{\mathrm{abs}}(X,F)\ ,\\ H^p\left(\Omega^\bullet_{\mathrm{rel}}(X,F),d^F\right) &\simeq H^p_{\mathrm{rel}}(X,F)\ . \end{aligned} \tag{1.1.13}$$

1.1.3. *Mayer-Vietoris exact sequence.*

Let Z be a closed $\mathscr{C}^\infty$-manifold. Let $i: Y \hookrightarrow Z$ be a compact hypersurface cutting Z into two pieces, denoted by Z_1 and Z_2. Then $Z = Z_1 \cup_Y Z_2$. Let $F \to Z$ be a complex vector bundle equipped with a flat connection ∇^F. We equip ∂Z_1 (resp. ∂Z_2) with relative (resp. absolute) boundary condition. All the notations and results developed in the previous subsections can be applied to $(Z_1, F|_{Z_1}, \nabla^F|_{Z_1})$ and $(Z_2, F|_{Z_2}, \nabla^F|_{Z_2})$.

Let K_{Z_1}, K_{Z_2} be smooth triangulations of Z_1, Z_2. Let K_Y be a smooth triangulation of Y, such that $K_Y = K_{Z_1}\cap Y = K_{Z_2}\cap Y$. Set

$$K_Z = (K_{Z_1}\backslash K_Y)\cup(K_{Z_2}\backslash K_Y)\cup K_Y\ , \tag{1.1.14}$$

which is a smooth triangulation of Z.

We have the following short exact sequence,

$$0 \longrightarrow (C^\bullet(K_{Z_1}/K_Y,F),\widetilde{\partial}) \longrightarrow (C^\bullet(K_Z,F),\widetilde{\partial}) \longrightarrow (C^\bullet(K_{Z_2},F),\widetilde{\partial}) \longrightarrow 0\ . \tag{1.1.15}$$

It induces a long exact sequence

$$\cdots \longrightarrow H^p_{\mathbf{bd}}(Z_1,F) \xrightarrow{\alpha_p} H^p(Z,F) \xrightarrow{\beta_p} H^p_{\mathbf{bd}}(Z_2,F) \xrightarrow{\delta_p} \cdots\ . \tag{1.1.16}$$

If we equip Z with a Riemannian metric g^{TZ} and F with a Hermitian metric h^F. By (1.1.13) and (1.1.16), we get a long exact sequence

$$\cdots \longrightarrow H^p\left(\Omega^\bullet_{\mathbf{bd}}(Z_1,F),d^F\right) \xrightarrow{\alpha_p} H^p\left(\Omega^\bullet(Z,F),d^F\right) \xrightarrow{\beta_p} H^p\left(\Omega^\bullet_{\mathbf{bd}}(Z_2,F),d^F\right) \xrightarrow{\delta_p} \cdots\ . \tag{1.1.17}$$

Proposition 1.1.2. *The maps* α_p, β_p *and* δ_p *in* (1.1.17) *are as follows.*

- *Let* $[\sigma]\in H^p\left(\Omega^\bullet_{\mathbf{bd}}(Z_1,F),d^F\right)$. *There exists* $\sigma'\in[\sigma]$ *which vanishes on a neighborhood of* Y. *Extending* σ' *by zero, we get* $\sigma''\in\Omega^p(Z,F)$. *We have* $\alpha_p([\sigma])=[\sigma'']$.
- *Let* $[\sigma]\in H^p\left(\Omega^\bullet(Z,F),d^F\right)$. *There exists* $\sigma'\in[\sigma]$ *such that* $\sigma'' := \sigma'\big|_{Z_2}\in\Omega^\bullet_{\mathbf{bd}}(Z_2,F)$. *We have* $\beta_p([\sigma])=[\sigma'']$.
- *Let* $[\sigma]\in H^p\left(\Omega^\bullet_{\mathbf{bd}}(Z_2,F),d^F\right)$. *There exists* $\sigma'\in\Omega^\bullet(Z,F)$ *such that* $\sigma'\big|_{Z_2}\in[\sigma]$. *Set* $\sigma''=d^F\sigma'\big|_{Z_1}$. *We have* $\delta_p([\sigma])=[\sigma'']$.

Let D^F_Z be the Hodge-de Rham operator on $\Omega^\bullet(Z,F)$. Let $D^F_{Z_j}$ $(j=1,2)$ be the Hodge-de Rham operator on $\Omega^\bullet_{\mathbf{bd}}(Z_j,F)$. Set

$$\mathscr{H}^\bullet(Z,F)=\ker D^F_Z\ ,\quad \mathscr{H}^\bullet_{\mathbf{bd}}(Z_j,F)=\ker D^F_{Z_j}\ ,\quad \text{for } j=1,2\ . \tag{1.1.18}$$

Applying Theorem 1.1.1, (1.1.16) induces the following long exact sequence,

$$\longrightarrow \mathscr{H}^p_{\mathbf{bd}}(Z_1,F)\xrightarrow{\alpha_p}\mathscr{H}^p(Z,F)\xrightarrow{\beta_p}\mathscr{H}^p_{\mathbf{bd}}(Z_2,F)\xrightarrow{\delta_p}\ . \tag{1.1.19}$$

We recall that $\langle\cdot,\cdot\rangle.$ is defined in §1.0.4.

The following proposition is a consequence of Theorem 1.1.1 and Proposition 1.1.2.

Proposition 1.1.3. *For $\omega\in\mathscr{H}^p_{\mathbf{bd}}(Z_1,F)$ and $\mu\in\mathscr{H}^p(Z,F)$, we have*

$$\big\langle\alpha_p(\omega),\mu\big\rangle_Z=\big\langle\omega,\mu\big\rangle_{Z_1}\ . \tag{1.1.20}$$

For $\omega\in\mathscr{H}^p(Z,F)$ and $\mu\in\mathscr{H}^p_{\mathbf{bd}}(Z_2,F)$, we have

$$\big\langle\beta_p(\omega),\mu\big\rangle_{Z_2}=\big\langle\omega,\mu\big\rangle_{Z_2}\ . \tag{1.1.21}$$

For $\omega\in\mathscr{H}^p_{\mathbf{bd}}(Z_2,F)$ and $\mu\in\mathscr{H}^{p+1}_{\mathbf{bd}}(Z_{1,R},F)$, we have

$$\big\langle\delta_p(\omega),\mu\big\rangle_{Z_1}=\big\langle\omega,i_{\frac{\partial}{\partial u}}\mu\big\rangle_Y\ . \tag{1.1.22}$$

1.2. **Hodge-de Rham operators on manifolds with cylindrical ends.**

Let Z_∞ be a Riemannian manifold with cylindrical ends, i.e., there exist a closed Riemannian manifold Y and an isometric inclusion $\mathbb{R}_+\times Y\subseteq Z_\infty$ such that $Z_\infty\backslash\big(\mathbb{R}_+\times Y\big)$ is compact. In this section, we review some properties of the Hodge Laplacian on Z_∞.

In §1.2.1, we consider the Hodge-de Rham operator acting on a closed manifold together with an additional odd Grassmannian variable du. In later subsections, u will serve as the coordinate on $\mathbb{R}_+$. In §1.2.2, we study the eigensections of the Hodge-de Rham operator acting on $I\times Y$, where I is a bounded open interval. In §1.2.3, we study the generalized eigensections of the Hodge-de Rham operator acting on Z_∞. In particular, (following [M94]) we define the scattering matrix and link it to the generalized eigensections. In §1.2.4, we study the generalized eigensections associated with the eigenvalue 0.

1.2.1. *Hodge-de Rham operator with an additional odd Grassmannian variable.*

Let Y be a closed $\mathscr{C}^\infty$-manifold. Let (F,∇^F) be a flat complex vector bundle over Y. Let g^{TY} be a Riemannian metric on Y. Let h^F be a Hermitian metric on F. Let D^F_Y be the Hodge-de Rham operator (cf. §1.0.4) acting on $\Omega^\bullet(Y,F)$.

Set

$$\mathscr{H}^\bullet(Y,F)=\ker D^{F,2}_Y\ . \tag{1.2.1}$$

For $\mu\in\mathbb{R}$, let $\mathscr{E}_\mu(Y,F)$ be the eigenspace of D^F_Y associated with the eigenvalue μ.

Let du be an additional odd Grassmannian variable, such that $(du)^2=0$. Let $\Omega^\bullet(Y,F[du])$ be the algebra generated by $\Omega^\bullet(Y,F)$ and du, i.e.,

$$\Omega^\bullet(Y,F[du])=\Omega^\bullet(Y,F)\oplus\Omega^\bullet(Y,F)du\ . \tag{1.2.2}$$

We equip $\Omega^\bullet(Y,F[du])$ with a grading : the degree p component is $\Omega^p(Y,F)\oplus\Omega^{p-1}(Y,F)du$.

The L^2-norm $\|\cdot\|_Y$ and its associated scalar product $\langle\cdot,\cdot\rangle_Y$ on $\Omega^\bullet(Y,F)$ (cf. §1.0.4) extend to $\Omega^\bullet(Y,F[du])$: for any $\tau_0,\tau_1\in\Omega^\bullet(Y,F)$,

$$\|\tau_0+du\wedge\tau_1\|^2_Y=\|\tau_0\|^2_Y+\|\tau_1\|^2_Y\ . \tag{1.2.3}$$

We define the actions $du\wedge$, $i_{\frac{\partial}{\partial u}}$ and $c(\frac{\partial}{\partial u})$ on $\Omega^\bullet(Y,F[du])$ as follows, for $\tau_0,\tau_1\in\Omega^\bullet(Y,F)$,

$$du\wedge(\tau_0+du\wedge\tau_1)=du\wedge\tau_0\ ,\qquad i_{\frac{\partial}{\partial u}}(\tau_0+du\wedge\tau_1)=\tau_1\ ,\qquad c(\tfrac{\partial}{\partial u})=du\wedge-i_{\frac{\partial}{\partial u}}\ . \tag{1.2.4}$$

The action of D_Y^F on $\Omega^\bullet(Y,F)$ extends to $\Omega^\bullet(Y,F[du])$ as follows,

$$D_Y^F(du\wedge\tau)=-du\wedge D_Y^F\tau\ ,\qquad \text{for } \tau\in\Omega^\bullet(Y,F)\ . \tag{1.2.5}$$

Let $\mathscr{H}^\bullet(Y,F[du])$ be the kernel of the extended action. Let $\mathscr{E}_\mu(Y,F[du])$ be the eigenspace of the extended action associated with the eigenvalue μ. We have

$$\begin{aligned}\mathscr{H}^\bullet(Y,F[du])&=\mathscr{H}^\bullet(Y,F)\oplus\mathscr{H}^\bullet(Y,F)du\ ,\\ \mathscr{E}_\mu(Y,F[du])&=\mathscr{E}_\mu(Y,F)\oplus\mathscr{E}_{-\mu}(Y,F)du\ .\end{aligned} \tag{1.2.6}$$

We remark that $c(\frac{\partial}{\partial u})$ exchanges $\mathscr{E}_{\pm\mu}(Y,F[du])$.

1.2.2. *Hodge-de Rham operator on a cylinder.*

Set $I=]a,b[\subseteq\mathbb{R}$. We consider the cylinder $I\times Y$. Let (u,y) $(u\in I, y\in Y)$ be the coordinates. Let $\pi_Y: I\times Y\to Y$ be the natural projection. We equip $I\times Y$ with the product metric (cf. (1.0.1)).

The pull back of F by π_Y is a flat vector bundle over $I\times Y$, which is still denoted by F. Its flat connection is defined by

$$\nabla^F=du\wedge\frac{\partial}{\partial u}+\nabla^F\big|_Y\ . \tag{1.2.7}$$

The pull back metric on F is still denoted h^F.

We have the canonical identification

$$\Omega^\bullet(I\times Y,F)\simeq\mathscr{C}^\infty\big(I,\Omega^\bullet(Y,F[du])\big)\ . \tag{1.2.8}$$

For $\omega\in\Omega^\bullet(I\times Y,F)$, $u\in I$, let $\omega_u\in\Omega^\bullet(Y,F[du])$ be the value of the corresponding function at u. For $\tau\in\Omega^\bullet(Y,F[du])$, let $\pi_Y^*\tau\in\Omega^\bullet(I\times Y,F)$ be the differential form corresponding to the constant function τ on I. For any $\omega,\omega'\in\Omega^\bullet(I\times Y,F)$, we have

$$\langle\omega,\omega'\rangle_{I\times Y}=\int_I\langle\omega_u,\omega'_u\rangle_Y du\ . \tag{1.2.9}$$

Let D_{IY}^F be the Hodge-de Rham operator acting on $\Omega^\bullet(I\times Y,F)$. We have

$$D_{IY}^F=c(\tfrac{\partial}{\partial u})\frac{\partial}{\partial u}+D_Y^F\ . \tag{1.2.10}$$

By the Green Formula, for $\omega_1,\omega_2\in\Omega^\bullet(I\times Y,F)$, we have

$$\big\langle D_{IY}^F\omega_1,\omega_2\big\rangle_{I\times Y}-\big\langle\omega_1,D_{IY}^F\omega_2\big\rangle_{I\times Y}=\big\langle c(\tfrac{\partial}{\partial u})\omega_{1,b},\omega_{2,b}\big\rangle_Y-\big\langle c(\tfrac{\partial}{\partial u})\omega_{1,a},\omega_{2,a}\big\rangle_Y\ . \tag{1.2.11}$$

Set

$$\delta_Y=\min\big\{|\mu|\ :\ \mu\in\mathrm{Sp}(D_Y^F)\backslash\{0\}\big\}\ . \tag{1.2.12}$$

Let $\omega\in\Omega^\bullet(I\times Y,F)$ such that $D_{IY}^F\omega=\lambda\omega$ with $|\lambda|<\delta_Y$. A direct calculation yields

$$\begin{aligned}\omega=\ &e^{-iu\lambda}\big(\phi_0^- - ic(\tfrac{\partial}{\partial u})\phi_0^-\big)+e^{iu\lambda}\big(\phi_0^+ + ic(\tfrac{\partial}{\partial u})\phi_0^+\big)\\ &+\sum_{\mu\neq0}\Big\{e^{-\sqrt{\mu^2-\lambda^2}u}\big(\phi_\mu^- - \frac{\mu-\lambda}{\sqrt{\mu^2-\lambda^2}}c(\tfrac{\partial}{\partial u})\phi_\mu^-\big)\\ &\qquad+e^{\sqrt{\mu^2-\lambda^2}u}\big(\phi_\mu^+ + \frac{\mu-\lambda}{\sqrt{\mu^2-\lambda^2}}c(\tfrac{\partial}{\partial u})\phi_\mu^+\big)\Big\}\ ,\end{aligned} \tag{1.2.13}$$

where $\mu \in \mathrm{Sp}(D_Y^F)$, $\phi_0^\pm \in \mathscr{H}^\bullet(Y,F)$, $\phi_\mu^\pm \in \mathscr{E}_\mu^\bullet(Y,F[du])$ (as convention, $\phi_\mu^\pm = 0$ for $\mu \notin \mathrm{Sp}(D_Y^F)$). Set

$$\omega^{\mathrm{zm},\pm} = e^{\pm iu\lambda}\big(\phi_0^\pm \pm ic(\tfrac{\partial}{\partial u})\phi_0^\pm\big) \ , \quad \omega^{\mathrm{zm}} = \omega^{\mathrm{zm},-} + \omega^{\mathrm{zm},+} \ . \tag{1.2.14}$$

The ω^{zm} is called the zeromode of ω. Set

$$\begin{aligned} &\omega^{\mu,\pm} = e^{\pm\sqrt{\mu^2-\lambda^2}u}\left(\phi_\mu^\pm \pm \frac{\mu-\lambda}{\sqrt{\mu^2-\lambda^2}}c(\tfrac{\partial}{\partial u})\phi_\mu^\pm\right) \ , \quad \omega^\mu = \omega^{\mu,-}+\omega^{\mu,+} \ , \\ &\omega^\pm = \sum_{\mu\neq 0}\omega^{\mu,\pm} \ , \quad \omega^{\mathrm{nz}} = \omega^- + \omega^+ \ . \end{aligned} \tag{1.2.15}$$

We have the following decomposition

$$\omega = \omega^{\mathrm{zm}} + \omega^{\mathrm{nz}} = \omega^{\mathrm{zm}} + \sum_{\mu\neq 0}(\omega^{\mu,+}+\omega^{\mu,-}) \ . \tag{1.2.16}$$

Furthermore, the above decomposition is fiberwise orthogonal, i.e., for $u \in I$, and $\mu' \neq \mu$, we have

$$\big\langle \omega_u^{\mathrm{zm}}, \omega_u^{\mu,+}+\omega_u^{\mu,-}\big\rangle_Y = 0 \ , \quad \big\langle \omega_u^{\mu,+}+\omega_u^{\mu,-}, \omega_u^{\mu',+}+\omega_u^{\mu',-}\big\rangle_Y = 0 \ . \tag{1.2.17}$$

For $a < u < v < b$, a simple estimate yields

$$\|\omega_v^-\|_Y \leqslant e^{-(v-u)\sqrt{\delta_Y^2-\lambda^2}}\|\omega_u^-\|_Y \ , \quad \|\omega_u^+\|_Y \leqslant e^{-(v-u)\sqrt{\delta_Y^2-\lambda^2}}\|\omega_v^+\|_Y \ . \tag{1.2.18}$$

By (1.2.4) and (1.2.14), $\|\omega_u^{\mathrm{zm}}\|_Y$ does not depend on $u \in I$. We denote

$$\|\omega^{\mathrm{zm}}\|_Y = \|\omega_u^{\mathrm{zm}}\|_Y \ . \tag{1.2.19}$$

Lemma 1.2.1. *For eigensections $\omega_1, \omega_2 \in \Omega^\bullet(I\times Y, F)$ with eigenvalue $\lambda \in]-\delta_Y, \delta_Y[$, we have*

$$\begin{aligned} &\langle \omega_1^{\mathrm{nz}}, \omega_2^{\mathrm{nz}}\rangle_{I\times Y} \leqslant \Big(\frac{1}{1-e^{-\sqrt{\delta_Y^2-\lambda^2}(b-a)}}\Big)^2 \cdot \frac{1}{\sqrt{\delta_Y^2-\lambda^2}} \cdot \|\omega_1\|_{\partial(I\times Y)} \cdot \|\omega_2\|_{\partial(I\times Y)} \ , \\ &\langle \omega_1^{\mathrm{zm}}, \omega_2^{\mathrm{zm}}\rangle_Y \leqslant \frac{1}{2}\|\omega_1\|_{\partial(I\times Y)} \cdot \|\omega_2\|_{\partial(I\times Y)} \ . \end{aligned} \tag{1.2.20}$$

Proof. The first inequality in (1.2.20) comes from (1.2.9), (1.2.12), (1.2.15), (1.2.17) and Cauchy-Schwarz inequality. The second inequality in (1.2.20) comes from (1.2.19). □

1.2.3. *Spectrum of Hodge-de Rham operators on manifolds with cylindrical ends.*

Let $(Z_\infty, g^{TZ_\infty})$ be a non-compact complet manifold with cylindrical end Y, i.e., there exists a subset $U \subseteq Z_\infty$ isometric to $\mathbb{R}_+ \times Y$ such that $Z_\infty \backslash U$ is compact.

Let (F, ∇^F) be a flat complex vector bundle over Z_∞. Using parallel transport along $\frac{\partial}{\partial u}$, $(F|_U, \nabla^F|_U)$ is identified with $\pi_Y^*(F|_Y, \nabla^F|_U)$, i.e., (1.2.7) holds. Let h^F be a Hermitian metric on F. We suppose that $(F|_U, h^F|_U)$ satisfies (1.0.1).

Let $D_{Z_\infty}^F$ be the Hodge-de Rham operator acting on $\Omega_{\mathrm{c}}^\bullet(Z_\infty, F)$. By [M94, Theorem 3.2], $D_{Z_\infty}^F$ is essentially self-adjoint. Its self-adjoint extension is still denoted by $D_{Z_\infty}^F$. Let $\Omega_{L^2}^\bullet(Z_\infty, F)$ be L^2-completion of $\Omega_{\mathrm{c}}^\bullet(Z_\infty, F)$, then

$$\Omega_{L^2}^\bullet(Z_\infty, F) = \mathscr{E}_{\mathrm{pp}}^\bullet(Z_\infty, F) \oplus \mathscr{E}_{\mathrm{sc}}^\bullet(Z_\infty, F) \oplus \mathscr{E}_{\mathrm{ac}}^\bullet(Z_\infty, F) \ , \tag{1.2.21}$$

where the vector spaces on the right hand side are, sequentially, associated with purely point (p.p.) spectrum, singularly continuous (s.c.) spectrum and absolutely continuous (a.c.) spectrum of $D_{Z_\infty}^F$ (cf. [RS80, chapter 7.2]). Let $D_{Z_\infty,\mathrm{pp}}^F$, $D_{Z_\infty,\mathrm{sc}}^F$ and $D_{Z_\infty,\mathrm{ac}}^F$ be the restriction of $D_{Z_\infty}^F$ to the corresponding vector subspaces.

For $\lambda \in \mathbb{R}$, let $\mathscr{E}_\lambda \subseteq \Omega^\bullet(Z_\infty, F)$ be the vector subspace of generalized eigensections of $D^F_{Z_\infty}$ associated with λ (cf. [Bere68, Chapter 5]). In this paper, it is sufficient to understand $\left(\mathscr{E}_\lambda\right)_{\lambda\in\mathbb{R}}$ as a family of vector subspaces of $\Omega^\bullet(Z_\infty, F)$ satisfying :

- for $\omega_\lambda \in \mathscr{E}_\lambda$, we have $D^F_{Z_\infty}\omega_\lambda = \lambda\omega_\lambda$;
- for $\omega \in \mathscr{E}^\bullet_{\mathrm{ac}}(Z_\infty, F) \cap \Omega^\bullet(Z_\infty, F)$, there exists a smooth family $\omega_\lambda \in \mathscr{E}_\lambda$, such that $\omega = \int_{\mathbb{R}} \omega_\lambda d\lambda$.

By definition, we have $\mathscr{E}_\lambda \cap \Omega^\bullet_{L^2}(Z_\infty, F) = 0$. As a consequence, a generalized eigensection is determined by its restriction to the cylinder.

On the cylinder, all the analysis done in §1.2.2 are still valid. We will continue to use the terminologies 'zeromode', 'non-zeromode', etc.

Before describing these $\mathscr{E}_\lambda$ in more detail, we need a model operator. We recall that $\Omega^\bullet(Y, F[du])$, $\mathscr{H}^\bullet(Y, F)$ and $\mathscr{E}_\mu(Y, F)$ are defined in §1.2.1. Let

$$\Pi : \Omega^\bullet(Y, F[du]) \to \mathscr{H}^\bullet(Y, F)du \oplus \bigoplus_{\mu>0} \Big((1-du)\mathscr{E}_\mu(Y, F) \oplus (1+du)\mathscr{E}_{-\mu}(Y, F)\Big) \tag{1.2.22}$$

be the orthogonal projection. We define the APS boundary condition ([APS75])

$$\Omega^\bullet_\Pi(\mathbb{R}_+ \times Y, F) = \Big\{\omega \in \Omega^\bullet(\mathbb{R}_+ \times Y, F) \ : \ \omega_0 \in \ker(\Pi)\Big\} \ , \tag{1.2.23}$$

where $\omega_0 = \omega_u\big|_{u=0} \in \Omega^\bullet(Y, F[du])$ is defined in §1.2.2. Let $D^F_{\mathbb{R}_+Y}$ be the Hodge-de Rham operator on $\mathbb{R}_+ \times Y$ with domain $\Omega^\bullet_\Pi(\mathbb{R}_+ \times Y, F)$. Then $D^F_{\mathbb{R}_+Y}$ only has a.c. spectrum.

Let $j : \mathbb{R}_+ \times Y \hookrightarrow Z_\infty$ be the canonical inclusion. Then j induces the inclusion

$$J : \Omega^\bullet_{L^2}(\mathbb{R}_+ \times Y, F) \hookrightarrow \Omega^\bullet_{L^2}(Z_\infty, F) \ . \tag{1.2.24}$$

We define the wave operators

$$W_\pm\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right) = \lim_{t\to\pm\infty} e^{itD^F_{Z_\infty}} J e^{-itD^F_{\mathbb{R}_+Y}} \ . \tag{1.2.25}$$

By [M94, Proposition 4.9], $W_\pm\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right)$ are well-defined.

The following theorem is established by Müller [M94, Theorem 4.1, Theorem 4.10].

Theorem 1.2.2. *The operator $D^F_{Z_\infty}$ has no singularly continuous spectrum.*

For $t > 0$, the operator $\exp\left(-tD^{F,2}_{Z_\infty,\mathrm{pp}}\right)$ is of trace class.

The wave operator $W_\pm\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right)$ gives a unitary equivalence between $D^F_{\mathbb{R}_+Y}$ and $D^F_{Z_\infty,\mathrm{ac}}$, i.e., $W_\pm\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right) : \Omega^\bullet_{L^2}(\mathbb{R}_+ \times Y, F) \to \mathscr{E}^\bullet_{\mathrm{ac}}(Z_\infty, F)$ is unitary, and the following diagram commutes,

$$\begin{tikzcd} \Omega^\bullet_{L^2}(\mathbb{R}_+ \times Y, F) \arrow[r, "D^F_{\mathbb{R}_+Y}"] \arrow[d, "W_\pm\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right)"'] & \Omega^\bullet_{L^2}(\mathbb{R}_+ \times Y, F) \arrow[d, "W_\pm\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right)"] \\ \mathscr{E}^\bullet_{\mathrm{ac}}(Z_\infty, F) \arrow[r, "D^F_{Z_\infty}"] & \mathscr{E}^\bullet_{\mathrm{ac}}(Z_\infty, F) \ . \end{tikzcd} \tag{1.2.26}$$

Set

$$C\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right) = W^*_+\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right) W_-\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right) \ , \tag{1.2.27}$$

which acts on $\Omega^\bullet_{L^2}(\mathbb{R}_+ \times Y, F)$. Then $C\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right)$ commutes with $D^F_{\mathbb{R}_+Y}$.

We remark that any generalized eigensection of $D^F_{\mathbb{R}_+Y}$ associated with $\lambda \in]-\delta_Y, \delta_Y[$ takes the following form,

$$E_0(\phi,\lambda) = e^{-i\lambda u}(\phi - ic(\tfrac{\partial}{\partial u})\phi) + e^{i\lambda u}(\phi + ic(\tfrac{\partial}{\partial u})\phi) \ , \tag{1.2.28}$$

where $\phi \in \mathscr{H}^\bullet(Y,F)$. Since $C\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right)$ commutes with $D^F_{\mathbb{R}_+Y}$, $C\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right)$ sends $E_0(\phi,\lambda)$ to $E_0(\phi',\lambda)$ for certain $\phi' \in \mathscr{H}^\bullet(Y,F)$.

Definition 1.2.3. For $\lambda \in]-\delta_Y, \delta_Y[$, let $C(\lambda) \in \mathrm{End}(\mathscr{H}^\bullet(Y,F))$ such that

$$C\left(D^F_{Z_\infty}, D^F_{\mathbb{R}_+Y}\right) E_0(\phi,\lambda) = E_0(C(\lambda)\phi,\lambda) \ . \tag{1.2.29}$$

We extend the action of $C(\lambda)$ to $\mathscr{H}^\bullet(Y,F[du])$ by demanding

$$C(\lambda)c(\tfrac{\partial}{\partial u}) = -c(\tfrac{\partial}{\partial u})C(\lambda) \ . \tag{1.2.30}$$

We call $C(\lambda)$ the scattering matrix associated with $D^F_{Z_\infty}$.

The following property is stated in [M94, §4].

Proposition 1.2.4. *Each generalized eigensection of $D^F_{Z_\infty,\mathrm{ac}}$ associated with $\lambda \in]-\delta_Y, \delta_Y[$ takes the following form over $\mathbb{R}_+ \times Y \simeq U \subseteq Z_\infty$:*

$$E(\phi,\lambda) = e^{-i\lambda u}(\phi - ic(\tfrac{\partial}{\partial u})\phi) + e^{i\lambda u}C(\lambda)(\phi - ic(\tfrac{\partial}{\partial u})\phi) + \theta(\phi,\lambda) \ , \tag{1.2.31}$$

where $\phi \in \mathscr{H}^\bullet(Y,F)$ and $\theta(\phi,\lambda) \in \Omega^\bullet_{L^2}(\mathbb{R}_+ \times Y, F)$. Furthermore, for $u \in \mathbb{R}_+$,

$$\theta_u(\phi,\lambda) \perp \mathscr{H}^\bullet(Y,F[du]) \ . \tag{1.2.32}$$

Conversely, for $\phi \in \mathscr{H}^\bullet(Y,F)$ and $\lambda \in]-\delta_Y, \delta_Y[$, there exists a unique generalized eigensection $E(\phi,\lambda)$ of $D^F_{Z_\infty,\mathrm{ac}}$ satisfying (1.2.31).

We remark that $E(\phi,\lambda)$ depends linearly on ϕ and analytically on λ (cf. [M94, §4]). Since $\mathscr{H}^\bullet(Y,F)$ is finite dimensional, there exists $C>0$ such that, for any $\phi \in \mathscr{H}^\bullet(Y,F)$ and $\lambda \in]-\delta_Y/2, \delta_Y/2[$, we have

$$\big\|E(\phi,\lambda)\big\|_{Z_\infty\setminus U} \leqslant C\big\|\phi\big\|_Y \ . \tag{1.2.33}$$

We list below several properties of $C(\lambda)$ (cf. [M94, §4]).

Proposition 1.2.5. *The following properties hold*

- *$C(\lambda)$ depends analytically on λ ;*
- *$C(\lambda) \in \mathrm{End}(\mathscr{H}^\bullet(Y,F[du]))$ is unitary ;*
- *$C(\lambda)$ preserves $\mathscr{H}^p(Y,F)$ and $\mathscr{H}^p(Y,F)du$ for any p ;*
- *$C(\lambda)C(-\lambda) = 1$, in particular, $C(0)^2 = 1$.*

1.2.4. *Extended L^2-solutions.*

Set

$$\mathscr{H}^\bullet_{L^2}(Z_\infty,F) = \Omega^\bullet_{L^2}(Z_\infty,F) \cap \ker\left(D^{F,2}_{Z_\infty}\right) \ , \tag{1.2.34}$$

The elements of $\mathscr{H}^\bullet_{L^2}(Z_\infty,F)$ are called L^2-solutions of $D^{F,2}_{Z_\infty}\omega = 0$.

We recall that the decomposition $\omega = \omega^{\mathrm{zm}} + \omega^{\mathrm{zn}} = \omega^{\mathrm{zm}} + \omega^- + \omega^+$ is given in (1.2.16).

Definition 1.2.6. Set

$$\mathscr{H}^\bullet(Z_\infty,F) = \Big\{(\omega,\hat\omega) \in \ker\left(D^{F,2}_{Z_\infty}\right) \oplus \mathscr{H}^\bullet(Y,F[du]) \ : \ \omega^+ = 0 \ , \omega^{\mathrm{zm}} = \pi_Y^*\hat\omega\Big\} \ , \tag{1.2.35}$$

The elements of $\mathscr{H}^\bullet(Z_\infty,F)$ are called extended L^2-solutions of $D^{F,2}_{Z_\infty}\omega = 0$.

Remark 1.2.7. In fact, $\mathscr{H}^\bullet(Z_\infty,F)$ is the vector subspace spanned by $\mathscr{H}^\bullet_{L^2}(Z_\infty,F)$ and generalized eigensections of $D^F_{Z_\infty}$ associated with 0, i.e.,

$$\mathscr{H}^\bullet(Z_\infty,F)=\mathscr{H}^\bullet_{L^2}(Z_\infty,F)\oplus\big\{E(\phi,0)\ :\ \phi\in\mathscr{H}^\bullet(Y,F)\big\}\ , \tag{1.2.36}$$

where $E(\phi,0)=E(\phi,\lambda)\big|_{\lambda=0}$ is given by (1.2.31).

Proposition 1.2.8. *For $(\omega,\hat{\omega})\in\mathscr{H}^\bullet(Z_\infty,F)$, we have*

$$d^F\omega=d^{F,*}\omega=0\ . \tag{1.2.37}$$

Proof. By (1.2.13), both $d^F\omega$ and $d^{F,*}\omega$ are L^2-sections, which are orthogonal with respect to the L^2-metric. Then $d^F\omega+d^{F,*}\omega=D^F\omega=0$ implies (1.2.37). □

Comparing (1.2.13) and Proposition 1.2.8, we get the following decomposition of $(\omega,\hat{\omega})\in\mathscr{H}^\bullet(Z_\infty,F)$ on the cylinder U,

$$\omega\big|_U=\pi_Y^*\hat{\omega}+\sum_{\mu>0\ ,\ \mu\in\mathrm{Sp}(D^F_Y)}e^{-\mu u}\big(\tau_{\mu,1}-du\wedge\tau_{\mu,2}\big)\ , \tag{1.2.38}$$

where $\tau_{\mu,1}\in\Omega^\bullet(Y,F)$, $\tau_{\mu,2}\in\Omega^{\bullet-1}(Y,F)$, and

$$d^F\tau_{\mu,1}=d^{F,*}\tau_{\mu,2}=0\ ,\quad d^{F,*}\tau_{\mu,1}=\mu\tau_{\mu,2}\ ,\quad d^F\tau_{\mu,2}=\mu\tau_{\mu,1}\ . \tag{1.2.39}$$

Definition 1.2.9. We define

$$\begin{aligned}\mathscr{R}_{d^F}&:\mathscr{H}^\bullet(Z_\infty,F)\to\Omega^{\bullet-1}(\mathbb{R}_+\times Y,F)\ ,\\ \mathscr{R}_{d^{F,*}}&:\mathscr{H}^\bullet(Z_\infty,F)\to\Omega^{\bullet+1}(\mathbb{R}_+\times Y,F)\ ,\end{aligned} \tag{1.2.40}$$

such that, for any $(\omega,\hat{\omega})\in\mathscr{H}^\bullet(Z_\infty,F)$, whose expansion is given by (1.2.38), we have

$$\mathscr{R}_{d^F}(\omega,\hat{\omega})=\sum_{\mu>0}\frac{1}{\mu}e^{-\mu u}\tau_{\mu,2}\ ,\quad \mathscr{R}_{d^{F,*}}(\omega,\hat{\omega})=\sum_{\mu>0}\frac{1}{\mu}e^{-\mu u}du\wedge\tau_{\mu,1}\ . \tag{1.2.41}$$

Proposition 1.2.10. *The following identities hold :*

$$\begin{aligned}d^F\mathscr{R}_{d^F}(\omega,\hat{\omega})&=\omega|_{\mathbb{R}_+\times Y}-\pi_Y^*\hat{\omega}\ ,\quad d^{F,*}\mathscr{R}_{d^F}(\omega,\hat{\omega})=0\ ,\\ d^{F,*}\mathscr{R}_{d^{F,*}}(\omega,\hat{\omega})&=\omega|_{\mathbb{R}_+\times Y}-\pi_Y^*\hat{\omega}\ ,\quad d^F\mathscr{R}_{d^{F,*}}(\omega,\hat{\omega})=0\ .\end{aligned} \tag{1.2.42}$$

Proof. These are direct consequences of (1.2.38), (1.2.39) and (1.2.41). □

Definition 1.2.11. Set

$$\mathscr{L}^\bullet=\Big\{\hat{\omega}\in\mathscr{H}^\bullet(Y,F[du])\ :\ \text{ there exists }\omega\text{ such that }(\omega,\hat{\omega})\in\mathscr{H}^\bullet(Z_\infty,F)\Big\}\ , \tag{1.2.43}$$

called the set of limiting values of $\mathscr{H}^\bullet(Z_\infty,F)$.

The scattering matrix associated with $D^F_{Z_\infty}$ is still denoted by $C(\lambda)$. Set $C=C(0)$. By (1.2.31), Proposition 1.2.5 and the fact that $\mathscr{L}^\bullet=\bigoplus\mathscr{L}^p$, we get

$$\mathscr{L}^\bullet=\mathrm{im}(C+1)=\ker(C-1)\ . \tag{1.2.44}$$

Let $P_{\mathscr{L}}:\mathscr{H}^\bullet(Y,F[du])\to\mathscr{L}^\bullet$ be the orthogonal projection. We have

$$C=2P_{\mathscr{L}}-1\ . \tag{1.2.45}$$

We recall that the operator $i_{\frac{\partial}{\partial u}}$ acting on $\mathscr{H}^\bullet(Y,F[du])$ is defined by (1.2.4). As consequences of (1.2.30), (1.2.44) and Proposition 1.2.5, there exist $\mathscr{L}^p_{\mathrm{abs}}\subseteq\mathscr{H}^p(Y,F)$ and $\mathscr{L}^p_{\mathrm{rel}}\subseteq\mathscr{H}^{p-1}(Y,F)du$ such that

$$\mathscr{L}^p=\mathscr{L}^p_{\mathrm{abs}}\oplus\mathscr{L}^p_{\mathrm{rel}}\ ,\quad \mathscr{L}^{p,\perp}_{\mathrm{abs}}=i_{\frac{\partial}{\partial u}}\mathscr{L}^{p+1}_{\mathrm{rel}}\ , \tag{1.2.46}$$

where $\mathscr{L}^{p,\perp}_{\mathrm{abs}} \subseteq \mathscr{H}^p(Y,F)$ is the orthogonal complement of $\mathscr{L}^p_{\mathrm{abs}}$. We call $\mathscr{L}^\bullet_{\mathrm{abs/rel}}$ the absolute/relative component of $\mathscr{L}^\bullet$.

We have the obvious short exact sequence

$$0 \longrightarrow \mathscr{H}^\bullet_{L^2}(Z_\infty,F) \longrightarrow \mathscr{H}^\bullet(Z_\infty,F) \longrightarrow \mathscr{L}^\bullet \longrightarrow 0\ . \tag{1.2.47}$$

We denote

$$\mathscr{H}^\bullet_{\mathrm{abs/rel}}(Z_\infty,F) = \Big\{(\omega,\hat{\omega}) \in \mathscr{H}^\bullet(Z_\infty,F)\ :\ \hat{\omega} \in \mathscr{L}^\bullet_{\mathrm{abs/rel}}\Big\}\ . \tag{1.2.48}$$

We get the following short exact sequence

$$0 \longrightarrow \mathscr{H}^\bullet_{L^2}(Z_\infty,F) \longrightarrow \mathscr{H}^\bullet_{\mathrm{abs/rel}}(Z_\infty,F) \longrightarrow \mathscr{L}^\bullet_{\mathrm{abs/rel}} \longrightarrow 0\ . \tag{1.2.49}$$

1.3. **Asymptotic properties of the spectrum.**

We recall that Z_R, F and $D^F_{Z_R}$ are defined in §1.0.2. In this section, we study the asymptotic behavior of $\mathrm{Sp}\left(D^F_{Z_R}\right)$ as $R \to \infty$.

In §1.3.1, we construct Z_R. In §1.3.2, we construct a model space of the eigensections of $D^F_{Z_R}$. In §1.3.3, we estimate the kernel of $D^{F,2}_{Z_R}$. In §1.3.4, we estimate the small eigenvalues of $D^F_{Z_R}$.

1.3.1. *Gluing of two manifolds with the same boundary.*

Let Z be a closed manifold. Let $i : Y \hookrightarrow Z$ be a compact hypersurface such that $Z\backslash Y = Z_1 \cup Z_2$ and $\partial Z_1 = \partial Z_2 = Y$. Then $Z = Z_1 \cup_Y Z_2$.

Let $U_j \subseteq Z_j$ $(j = 1,2)$ be a collar neighborhood of $\partial Z_j \simeq Y$. More precisely, we fix the diffeomorphisms

$$i_1 :]-1,0] \times Y \to U_1\ ,\quad i_2 :\quad [0,1[\times Y \to U_2\ , \tag{1.3.1}$$

such that $i_j(\{0\}\times Y) = \partial Z_j$ $(j=1,2)$. Set $U = U_1 \cup_Y U_2 \subseteq Z$. Then i_1 and i_2 induce the identification

$$i :]-1,1[\times Y \to U \subseteq Z\ . \tag{1.3.2}$$

Let (F,∇^F) be a flat vector bundle over Z.

Let g^{TZ} be a Riemannian metric on Z. Let h^F be a Hermitian metric on F. We suppose that (1.0.1) holds.

Set

$$\begin{aligned} &Z_{1,R} = Z_1 \cup_Y [0,R]\times Y\ ,\quad Z_{2,R} = Z_2 \cup_Y [-R,0]\times Y\ ,\quad \text{for } 0 \leqslant R \leqslant \infty\ ,\\ &Z_{1,\infty} = Z_1 \cup_Y [0,\infty[\times Y\ ,\quad Z_{2,\infty} = Z_2 \cup_Y]-\infty,0]\times Y\ , \end{aligned} \tag{1.3.3}$$

where the gluing identifies $\partial Z_j \simeq Y$ $(j=1,2)$ to $\{0\}\times Y$. For $0 \leqslant R < \infty$, we define

$$\begin{aligned} f_R : [0,2R]\times Y &\to [-2R,0]\times Y\\ (u,y) &\mapsto (u-2R,y)\ . \end{aligned} \tag{1.3.4}$$

Set

$$Z_R = Z_{1,2R} \cup_{f_R} Z_{2,2R} = Z_{1,R} \cup_Y Z_{2.R}\ . \tag{1.3.5}$$

Then (F,∇^F) extends to a flat vector bundle over Z_R such that (1.2.7) holds. Moreover, g^{TZ} and h^F extend to Z_R such that (1.0.1) holds.

In the sequel, all the canonical projections from $[-R,0]\times Y$, $[0,R]\times Y$ and $[-R,R]\times Y$ $(0 \leqslant R \leqslant \infty)$ onto Y will simply be denoted by π_Y if there is no confusion.

In the sequel, for $0 \leqslant R \leqslant \infty$, $[0,R]\times Y \subseteq Z_{1,R}$ (resp. $[-R,0]\times Y \subseteq Z_{2,R}$), the cylindrical part of $Z_{1,R}$ (resp. $Z_{2,R}$), will be refered to as $I_{1,R}Y$ (resp. $I_{2,R}Y$); if $R < \infty$, the cylindrical part of Z_R, i.e., the gluing of $I_{1,R}Y$ and $I_{2,R}Y$, will be refered to as I_RY. On $I_{1,R}Y$, we use the coordinates (u_1, y) with $u_1 \in [0, R]$, $y \in Y$; on $I_{2,R}Y$, we use the coordinates (u_2, y) with $u_2 \in [-R, 0]$, $y \in Y$; on I_RY, we use the coordinates (u, y) with $u \in [-R, R]$, $y \in Y$. Under the identifications $I_{1,2R}Y \simeq I_{2,2R}Y \simeq I_RY$ induced by (1.3.5), the transformation of the coordinates is given by

$$u = u_1 - R = u_2 + R \,. \tag{1.3.6}$$

For $A \subseteq \mathbb{R}$, set

$$\begin{aligned} I_{j,R}Y(A) &= \big\{ (u_j, y) \in I_{j,R}Y \ : \ u_j \in A \big\} \,, \quad \text{for } j = 1,2 \,, \\ I_RY(A) &= \big\{ (u, y) \in I_RY \ : \ u \in A \big\} \,. \end{aligned} \tag{1.3.7}$$

We will always use the following identifications : for $R' \leqslant R$,

$$Z_{j,R'} \subseteq Z_{j,R} \,, \quad \text{for } j = 1,2 \,, \tag{1.3.8}$$

which is the unique isometric inclusion fixing $Z_{j,0}$; for $R' \leqslant 2R$,

$$Z_{j,R'} \subseteq Z_{j,2R} \subseteq Z_R \,, \quad \text{for } j = 1,2 \,, \tag{1.3.9}$$

where the second inclusion is induced by (1.3.5).

Let $D^F_{Z_R}$ be the Hodge-de Rham operator acting on $\Omega^\bullet(Z_R, F)$ (see §1.0.4).

1.3.2. *Models of eigenspaces associated to small eigenvalues.*

Let $\mathscr{H}^\bullet_{L^2}(Z_{j,\infty}, F)$ and $\mathscr{H}^\bullet(Z_{j,\infty}, F)$ $(j = 1, 2)$ be as (1.2.34) and (1.2.35) with Z_∞ replaced by $Z_{j,\infty}$ and u replaced by u_j (cf. (1.3.1)). It is important to notice that $\frac{\partial}{\partial u_2}$ points to the inner side of Z_2. This is different from the choice in (1.2.35). Set

$$\begin{aligned} \mathscr{H}^\bullet(Z_{12,\infty}, F) = \Big\{ (\omega_1, \omega_2, \hat{\omega}) \ : \ & (\omega_1, \hat{\omega}) \in \mathscr{H}^\bullet(Z_{1,\infty}, F) \,, \\ & (\omega_2, \hat{\omega}) \in \mathscr{H}^\bullet(Z_{2,\infty}, F) \Big\} \,. \end{aligned} \tag{1.3.10}$$

Let $\mathscr{L}^\bullet_j \subseteq \mathscr{H}^\bullet(Y, F[du])$ $(j = 1, 2)$ be the set of limiting values of $\mathscr{H}^\bullet(Z_{j,\infty}, F)$ (cf. (1.2.43)). There is a natural injection

$$\begin{aligned} \mathscr{H}^\bullet_{L^2}(Z_{1,\infty}, F) \oplus \mathscr{H}^\bullet_{L^2}(Z_{2,\infty}, F) &\to \mathscr{H}^\bullet(Z_{12,\infty}, F) \\ (\omega_1, \omega_2) &\mapsto (\omega_1, \omega_2, 0) \,. \end{aligned} \tag{1.3.11}$$

There is a natural surjection

$$\begin{aligned} \mathscr{H}^\bullet(Z_{12,\infty}, F) &\to \mathscr{L}^\bullet_1 \cap \mathscr{L}^\bullet_2 \\ (\omega_1, \omega_2, \hat{\omega}) &\mapsto \hat{\omega} \,. \end{aligned} \tag{1.3.12}$$

We have the following short exact sequence,

$$0 \to \mathscr{H}^\bullet_{L^2}(Z_{1,\infty}, F) \oplus \mathscr{H}^\bullet_{L^2}(Z_{2,\infty}, F) \to \mathscr{H}^\bullet(Z_{12,\infty}, F) \to \mathscr{L}^\bullet_1 \cap \mathscr{L}^\bullet_2 \to 0 \,. \tag{1.3.13}$$

Recall that the L^2-norm $\|\cdot\|_\cdot$ is defined in §1.0.4. For $(\omega_1, \omega_2, \hat{\omega}) \in \mathscr{H}^\bullet(Z_{12,\infty}, F)$, set

$$\big\|(\omega_1, \omega_2, \hat{\omega})\big\|^2_{\mathscr{H}^\bullet(Z_{12,\infty}, F), R} = \|\omega_1\|^2_{Z_{1,R}} + \|\omega_2\|^2_{Z_{2,R}} \,. \tag{1.3.14}$$

We will drop the subscript R, if $R = 0$. By (1.2.20) and (1.2.35), there exists $C > 0$, such that, for any $(\omega_1, \omega_2, \hat{\omega}) \in \mathscr{H}^\bullet(Z_{12,\infty}, F)$,

$$\big\|(\omega_1, \omega_2, \hat{\omega})\big\|^2_{\mathscr{H}^\bullet(Z_{12,\infty}, F), R} \leqslant \big(1 + CR\big) \big\|(\omega_1, \omega_2, \hat{\omega})\big\|^2_{\mathscr{H}^\bullet(Z_{12,\infty}, F)} \,. \tag{1.3.15}$$

In the rest of this section, $\mathscr{H}^\bullet(Z_{12,\infty}, F)$ will serve as the model space of $\ker\big(D^{F,2}_{Z_R}\big)$.

Recall that δ_Y was defined in (1.2.12). For $\lambda \in]-\delta_Y, 0[\cup]0, \delta_Y[$, $j=1,2$, set

(1.3.16)
$$\mathscr{E}_\lambda(Z_{j,\infty},F) = \Big\{(\omega,\omega^{\mathrm{zm}}) \ : \ \omega \in \Omega^\bullet(Z_{j,\infty},F) \text{ is a generalized eigensection of } D^F_{Z_{j,\infty}} \text{ associated with } \lambda \,, \omega^{\mathrm{zm}} \in \Omega^\bullet(I_{j,\infty}Y,F) \text{ is the zeromode of } \omega\Big\} \,.$$

Recall that f_R is defined in (1.3.4). For $R>0$, set

$$\mathscr{E}_{\lambda,R}(Z_{12,\infty},F) = \Big\{(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}}) \ : \ (\omega_j,\omega_j^{\mathrm{zm}}) \in \mathscr{E}_\lambda(Z_{j,\infty},F) \ , \text{ for } j=1,2 \ , \ \omega_1^{\mathrm{zm}}\big|_{I_{1,\infty}Y([0,2R])} = f_R^*\big(\omega_2^{\mathrm{zm}}\big|_{I_{2,\infty}Y([-2R,0])}\big)\Big\} \,. \tag{1.3.17}$$

Let $C_j(\lambda) \in \mathrm{End}(\mathscr{H}^\bullet(Y,F[du]))$ $(j=1,2)$ be the scattering matrices associated with $D^F_{Z_{j,\infty}}$. For convenience, we take the following definition of scattering matrix : $C_j(\lambda)$ is the unique matrix such that (1.2.31) holds with u replaced by u_j (cf. (1.3.1)). Since $\frac{\partial}{\partial u_2}$ points to the inner side of Z_2, $C_2(\lambda)$ is the inverse of the scattering matrix in the sense of Definition 1.2.3. Set

$$C_{12}(\lambda) = C_2^{-1}(\lambda)C_1(\lambda) \in \mathrm{End}(\mathscr{H}^\bullet(Y,F[du])) \,. \tag{1.3.18}$$

For $R \geqslant 0$, set

$$\Lambda_R = \Big\{\lambda \in \mathbb{R} \ : \ \det\Big(e^{4i\lambda R}C_{12}(\lambda)\big|_{\mathscr{H}^\bullet(Y,F)} - 1\Big) = 0\Big\} \tag{1.3.19}$$

(counting multiplicity). By (1.2.31), (1.3.4) and (1.3.16), we have

$$\Big\{\lambda \in \mathbb{R} \ : \ \mathscr{E}_{\lambda,R}(Z_{12,\infty},F) \neq \{0\}\Big\} = \Lambda_R \,. \tag{1.3.20}$$

For $A \subseteq]-\delta_Y, 0[\cup]0, \delta_Y[$, set

$$\mathscr{E}_{A,R}(Z_{12,\infty},F) = \bigoplus_{\lambda\in A} \mathscr{E}_{\lambda,R}(Z_{12,\infty},F) \,. \tag{1.3.21}$$

For $(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}}) \in \mathscr{E}_{A,R}(Z_{12,\infty},F)$, set

$$\big\|(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}})\big\|^2_{\mathscr{E}_{A,R}(Z_{12,\infty},F)} = \big\|\omega_1\big\|^2_{Z_{1,0}} + \big\|\omega_2\big\|^2_{Z_{2,0}} \,. \tag{1.3.22}$$

In the rest of this section, $\mathscr{E}_{A,R}(Z_{12,\infty},F)$ will serve as the model space of the eigenspace of $D^F_{Z_R}$ with eigenvalues in A.

1.3.3. *Approximating the kernels.*

Let $\gamma \in \mathscr{C}^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\gamma \geqslant 0$, $\mathrm{supp}(\gamma) \subseteq [-\frac{1}{2},\frac{1}{2}]$ and $\int_{-\frac{1}{2}}^{\frac{1}{2}}\gamma(s)ds = 1$. We define $\chi_{2,1} \in \mathscr{C}^\infty([-1,1])$ by

$$\chi_{2,1}(u) = \begin{cases} 0 & \text{if } -1 \leqslant u < 0 \,, \\ \int_{-1}^{2u-1}\gamma(s)ds & \text{if } 0 \leqslant u \leqslant 1 \,. \end{cases} \tag{1.3.23}$$

Then $\chi_{2,1}(u) = 1$ for $u > \frac{3}{4}$. For $j=1,2$, we define $\chi_{j,R} \in \mathscr{C}^\infty([-R,R])$ by

$$\chi_{j,R}(u) = \chi_{2,1}\big((-1)^j u/R\big) \,. \tag{1.3.24}$$

We may view $\chi_{j,R}$ as a function on I_RY, i.e., for $(u,y) \in I_RY$, $\chi_{j,R}(u,y) = \chi_{j,R}(u)$.

We recall that the following maps are defined in Definition 1.2.9,

$$\mathscr{R}_{d^F}, \mathscr{R}_{d^{F,*}} : \mathscr{H}^\bullet(Z_{j,\infty},F) \to \Omega^\bullet(I_{j,\infty}Y,F) \,, \quad \text{for } j=1,2 \,. \tag{1.3.25}$$

Composing the identification $I_RY \simeq I_{j,2R}Y$ ($j = 1,2$) induced by (1.3.9) and the injection $I_{j,2R}Y \subseteq I_{j,\infty}Y$ induced by (1.3.8), we get $I_RY \hookrightarrow I_{j,\infty}Y$, which induces

$$\Omega^\bullet(I_{j,\infty}Y, F) \to \Omega^\bullet(I_RY, F) . \tag{1.3.26}$$

Composing (1.3.25) and (1.3.26), we get

$$\mathscr{R}_{d^F,j}\ ,\mathscr{R}_{d^{F,*},j} :\ \mathscr{H}^\bullet(Z_{j,\infty}, F) \to \Omega^\bullet(I_RY, F)\ , \quad \text{for } j = 1,2\ . \tag{1.3.27}$$

Definition 1.3.1. We define

$$F_{Z_R}\ ,\ G_{Z_R}\ :\ \mathscr{H}^\bullet(Z_{12,\infty}, F) \to \Omega^\bullet(Z_R, F) \tag{1.3.28}$$

as follows: for $(\omega_1, \omega_2, \hat\omega) \in \mathscr{H}^\bullet(Z_{12,\infty}, F)$,

$$\begin{aligned} F_{Z_R}(\omega_1,\omega_2,\hat\omega)\big|_{Z_{j,0}} &= G_{Z_R}(\omega_1,\omega_2,\hat\omega)\big|_{Z_{j,0}} = \omega_j\ , \quad \text{for } j = 1,2\ ,\\ F_{Z_R}(\omega_1,\omega_2,\hat\omega)\big|_{I_RY} &= \pi_Y^*\hat\omega + \sum_{j=1}^2 d^F\Big(\chi_{j,R}\,\mathscr{R}_{d^F,j}(\omega_j,\hat\omega)\Big)\ ,\\ G_{Z_R}(\omega_1,\omega_2,\hat\omega)\big|_{I_RY} &= \pi_Y^*\hat\omega + \sum_{j=1}^2 d^{F,*}\Big(\chi_{j,R}\,\mathscr{R}_{d^{F,*},j}(\omega_j,\hat\omega)\Big)\ . \end{aligned} \tag{1.3.29}$$

By (1.2.42), F_{Z_R} and G_{Z_R} are well-defined. Furthermore, we have

$$d^F F_{Z_R}(\omega_1,\omega_2,\hat\omega) = d^{F,*}G_{Z_R}(\omega_1,\omega_2,\hat\omega) = 0\ . \tag{1.3.30}$$

Remark 1.3.2. This gluing technique was initiated by Atiyah-Patodi-Singer [APS75]. They glued ω_1 and ω_2 directly using partitions of unity. The difference between the standard Atiyah-Patodi-Singer gluing and ours is $\mathscr{O}(e^{-cR})$-small as $R \to \infty$.

We recall that $U_j \subseteq Z_j$ ($j = 1,2$) is a neighborhood of $Y = \partial Z_j$. Gluing the identifications $U_1 =]-1,0] \times Y$, $I_RY = [-R,R] \times Y$, $U_2 = [0,1[\times Y$ by shifting the coordinates, we get the identification $U_1 \cup I_RY \cup U_2 =]-R-1, R+1[\times Y$. Let $\phi_R :]-R-1, R+1[\to]-1,1[$ be a smooth function such that

$$\phi(-u) = -\phi(u)\ , \phi'(u) > 0\ ,\ \phi_R(u) = u + R \ \text{ for } u \in [-R-1, -R-1/2]\ . \tag{1.3.31}$$

We define a diffeomorphism $\varphi_R : Z_R \to Z$ as follows:

$$\begin{aligned} &\varphi_R\big|_{Z_j\backslash U_j} = \mathrm{Id}_{Z_j\backslash U_j}\ , \quad \text{for } j = 1,2\ ,\\ &\varphi_R(u,y) = (\phi_R(u), y) \in U_1 \cup U_2 \subseteq Z \quad \text{for } (u,y) \in U_1 \cup I_RY \cup U_2 \subseteq Z_R\ . \end{aligned} \tag{1.3.32}$$

Then φ_R induces the canonical isomorphism $H^\bullet(Z_R, Y) \simeq H^\bullet(Z, F)$.

Proposition 1.3.3. *For $R' > R > 1$, $(\omega_1,\omega_2,\hat\omega) \in \mathscr{H}^\bullet(Z_{12,\infty}, F)$ with $\hat\omega \in \mathscr{H}^\bullet(Y, F)$,*

$$\big[F_{Z_R}(\omega_1,\omega_2,\hat\omega)\big] = \big[F_{Z_{R'}}(\omega_1,\omega_2,\hat\omega)\big] \in H^\bullet(Z, F)\ . \tag{1.3.33}$$

Proof. By inserting enough numbers between R and R', we may assume that $R'/R \leqslant 7/6$.

We define $\tilde\phi_{R,R'} : [-R,R] \to [-R', R']$ by

$$\tilde\phi_{R,R'}(u) = \begin{cases} u - R' + R & \text{if } u \in [-R, -\frac{1}{8}R]\ ,\\ u - (R'-R)\chi_{1,R/8}(u) & \text{if } u \in [-\frac{1}{8}R, 0]\ ,\\ u + (R'-R)\chi_{2,R/8}(u) & \text{if } u \in [0, \frac{1}{8}R]\ ,\\ u + R' - R & \text{if } u \in [\frac{1}{8}R, R]\ . \end{cases} \tag{1.3.34}$$

We construct a diffeomorphism $\tilde{\varphi}_{R,R'} : Z_R \to Z_{R'}$ as follows: the restriction of $\tilde{\varphi}_{R,R'}$ to $Z_{1,0} \cup Z_{2,0} \simeq Z_R \backslash I_R Y \simeq Z_{R'} \backslash I_{R'} Y$ is the identity map, for any $(u,y) \in I_R Y$, $\tilde{\varphi}_{R,R'}(u,y) = (\tilde{\phi}_{R,R'}(u), y) \in I_{R'} Y$. Then $\tilde{\varphi}_{R,R'}$ is homotopic to $\varphi_{R'}^{-1} \circ \varphi_R$.

Let $\mu \in \Omega^\bullet(Z_R, F)$ such that

$$\mu\,\big|_{Z_R \backslash I_R Y} = 0\ , \quad \mu\,\big|_{I_R Y} = \sum_{j=1}^{2} (\chi_{j,R} - \chi_{j,R'})\, \mathscr{R}_{d^F,j}(\omega_j, \hat{\omega})\ . \tag{1.3.35}$$

By (1.3.29), (1.3.34) and (1.3.35), we have

$$F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \tilde{\varphi}_{R,R'}^* F_{Z_{R'}}(\omega_1, \omega_2, \hat{\omega}) = d^F \mu\ . \tag{1.3.36}$$

Taking the cohomology class of (1.3.36), we terminate the proof. $\square$

We recall that $\|\cdot\|_\cdot$ is defined by (1.0.22) and $\|\cdot\|_{\mathscr{H}^\bullet(Z_{12,\infty},F),R}$ is defined by (1.3.14).

Proposition 1.3.4. *There exist $c > 0$, $R_0 > 0$, such that, for $R > R_0$, $(\omega_1, \omega_2, \hat{\omega}) \in \mathscr{H}^\bullet(Z_{12,\infty}, F)$, we have*

$$1 - e^{-cR} \leqslant \frac{\|F_{Z_R}(\omega_1, \omega_2, \hat{\omega})\|_{Z_R}}{\|(\omega_1, \omega_2, \hat{\omega})\|_{\mathscr{H}^\bullet(Z_{12,\infty},F),R}} \leqslant 1 + e^{-cR}\ . \tag{1.3.37}$$

Proof. It is sufficient to show that

$$\|F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_j\|_{Z_{j,R}} \leqslant e^{-cR} \|\omega_1\|_{Z_{j,0}}\ , \qquad \text{for } j = 1, 2\ . \tag{1.3.38}$$

We will only show the case $j = 1$.

By our construction, $F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_1$ vanishes on $Z_{1,0}$. By (1.2.42), we have

$$\begin{aligned} &\big(F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_1\big)\big|_{I_{1,R} Y} \\ &= d^F\big(\chi_{1,R}\, \mathscr{R}_{d^F,1}(\omega_1, \hat{\omega})\big) + \pi_Y^* \hat{\omega} - \omega_1 \\ &= \left(\frac{\partial}{\partial u} \chi_{1,R}\right) du \wedge \mathscr{R}_{d^F,1}(\omega_1, \hat{\omega}) + (\chi_{1,R} - 1)\,(\omega_1 - \pi_Y^* \hat{\omega})\ . \end{aligned} \tag{1.3.39}$$

By the definition of $\chi_{1,R}$, $\frac{\partial}{\partial u} \chi_{1,R}$ is bounded by 1 and with support in $I_R Y([-\frac{3}{4}R, -\frac{1}{4}R])$; $\chi_{1,R} - 1$ is bounded by 1 and with support in $I_R Y([-\frac{3}{4}R, 0])$. Then

$$\begin{aligned} &\|F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_1\|_{Z_{1,R}} \\ &\leqslant \big\|\mathscr{R}_{d^F,1}(\omega_1, \hat{\omega})\big\|_{I_R Y([-\frac{3}{4}R, -\frac{1}{4}R])} + \|\omega_1 - \pi_Y^* \hat{\omega}\|_{I_R Y([-\frac{3}{4}R, 0])}\ . \end{aligned} \tag{1.3.40}$$

By Definition 1.2.9, we have

$$\big\|\mathscr{R}_{d^F,1}(\omega_1, \hat{\omega})\big\|^2_{I_R Y([-\frac{3}{4}R, -\frac{1}{4}R])} \leqslant \delta_Y^{-2} e^{-\frac{1}{2}\delta_Y R} \|\omega_1\|^2_{\partial Z_{1,0}}\ . \tag{1.3.41}$$

By Lemma 1.2.1, (1.2.18) and (1.2.19), we have

$$\begin{aligned} \|\omega_1 - \pi_Y^* \hat{\omega}\|^2_{I_R Y([-\frac{3}{4}R, 0])} &\leqslant \left(\frac{1}{1 - e^{-\frac{3}{4}\delta_Y R}}\right)^2 \cdot \delta_Y^{-1} \cdot \|\omega_1 - \pi_Y^* \hat{\omega}\|^2_{\partial Z_{1,R/4} \cup \partial Z_{1,R}} \\ &\leqslant \left(\frac{1}{1 - e^{-\frac{3}{4}\delta_Y R}}\right)^2 \cdot 2\delta_Y^{-1} e^{-\frac{1}{2}\delta_Y R} \|\omega_1\|^2_{\partial Z_{1,0}}\ . \end{aligned} \tag{1.3.42}$$

Comparing (1.3.40)-(1.3.42), it only rests to show that

$$\|\omega_1\|_{\partial Z_{1,0}} \leqslant C \|\omega_1\|_{Z_{1,0}}\ . \tag{1.3.43}$$

Let $\|\cdot\|_{1,Z_{1,0}}$ be the H^1-norm on $\mathscr{C}^\infty(Z_{1,0},F)$. We fix $\varepsilon>0$. By the ellipticity of the Hodge-de Rham operator, we may suppose that, for any $\omega\in\Omega^\bullet(Z_{1,\infty},F)$,

$$\|\omega\|^2_{1,Z_{1,0}}\leqslant\|\omega\|^2_{Z_{1,\varepsilon}}+\|D^F_{Z_{1,\infty}}\omega\|^2_{Z_{1,\varepsilon}}\ . \tag{1.3.44}$$

In particular,

$$\|\omega_1\|^2_{1,Z_{1,0}}\leqslant\|\omega_1\|^2_{Z_{1,\varepsilon}}\ . \tag{1.3.45}$$

By the trace theorem, there exists $C_2>0$, such that, for any ω_1, we have

$$\|\omega_1\|^2_{\partial Z_{1,0}}\leqslant C_2\|\omega_1\|^2_{1,Z_{1,0}}\ . \tag{1.3.46}$$

By (1.3.15), (1.3.45) and (1.3.46), we get (1.3.43). □

For going further, we need a uniform Sobolev inequality on Z_R for $R\geqslant 0$. Let $m\in\mathbb{N}$ such that $m>\frac{1}{2}\dim Z_R$. We recall that $\|\cdot\|_{\mathscr{C}^0,\cdot}$ is defined by (1.0.23).

Proposition 1.3.5. *There exists $C>0$ such that, for $R>0$, $\omega\in\Omega^\bullet(Z_R,F)$, we have*

$$\big\|\omega\big\|_{\mathscr{C}^0,Z_R}\leqslant C\left(\big\|\omega\big\|_{Z_R}+\big\|D^{F,m}_{Z_R}\omega\big\|_{Z_R}\right)\ . \tag{1.3.47}$$

Proof. By repeating the proof of the classical Sobolev inequality on each Z_R, we find that the constant C, which, a priori, depends on R, is uniformly bounded for $R\geqslant 0$. □

Let $P^{\ker(D^{F,2}_{Z_R})}:\Omega^\bullet(Z_R,F)\to\ker\left(D^{F,2}_{Z_R}\right)$ be the orthogonal projection.

Definition 1.3.6. Set

$$\mathscr{F}_{Z_R}=P^{\ker(D^{F,2}_{Z_R})}\circ F_{Z_R}\ ,\quad \mathscr{G}_{Z_R}=P^{\ker(D^{F,2}_{Z_R})}\circ G_{Z_R}\ . \tag{1.3.48}$$

Proposition 1.3.7. *There exist $c>0$, $R_0>0$ such that, for $R>R_0$, $(\omega_1,\omega_2,\hat\omega)\in\mathscr{H}^\bullet(Z_{12,\infty},F)$, we have*

$$\big\|(F_{Z_R}-\mathscr{F}_{Z_R})(\omega_1,\omega_2,\hat\omega)\big\|_{\mathscr{C}^0,Z_R}\leqslant e^{-cR}\big\|(\omega_1,\omega_2,\hat\omega)\big\|_{\mathscr{H}^\bullet(Z_{12,\infty},F)}\ . \tag{1.3.49}$$

As a consequence $\mathscr{F}_{Z_R}:\mathscr{H}^\bullet(Z_{12,\infty},F)\to\ker\left(D^{F,2}_{Z_R}\right)$ is injective for R large enough.

Proof. By (1.2.42) and (1.3.29), $\operatorname{supp}\big((F_{Z_R}-G_{Z_R})(\omega_1,\omega_2,\hat\omega)\big)\subseteq I_RY$, and

$$\begin{aligned}&\big(F_{Z_R}-G_{Z_R}\big)(\omega_1,\omega_2,\hat\omega)\big|_{I_RY}\\&=\sum_{j=1}^2\Big(\frac{\partial}{\partial u}\chi_{j,R}\Big)\Big(du\wedge\mathscr{R}_{d^F,j}(\omega_j,\hat\omega)+i_{\frac{\partial}{\partial u}}\mathscr{R}_{d^{F,*},j}(\omega_j,\hat\omega)\Big)\ .\end{aligned} \tag{1.3.50}$$

More generally, by (1.1.6), (1.2.42) and (1.3.50), for any $m\in\mathbb{N}$,

$$\begin{aligned}&D^{F,2m}_{Z_R}\big(F_{Z_R}-G_{Z_R}\big)(\omega_1,\omega_2,\hat\omega)\big|_{I_RY}\\&=(-1)^m\sum_{j=1}^2\left(\frac{\partial^{2m+1}}{\partial u^{2m+1}}\chi_{j,R}\right)\Big(du\wedge\mathscr{R}_{d^F,j}(\omega_j,\hat\omega)+i_{\frac{\partial}{\partial u}}\mathscr{R}_{d^{F,*},j}(\omega_j,\hat\omega)\Big)\ .\end{aligned} \tag{1.3.51}$$

Set

$$\alpha_m=\sup_{u\in[-1,1]}\left|\frac{\partial^m}{\partial u^m}\chi_{2,1}(u)\right|\ . \tag{1.3.52}$$

Since $\mathrm{supp}\big(\frac{\partial}{\partial u}\chi_{1,R}\big) \subseteq [-\frac{3}{4}R, -\frac{1}{4}R]$ and $\mathrm{supp}\big(\frac{\partial}{\partial u}\chi_{2,R}\big) \subseteq [\frac{1}{4}R, \frac{3}{4}R]$, we get

$$\begin{aligned}
&\Big\| D^{F,2m}_{Z_R}\left(F_{Z_R} - G_{Z_R}\right)(\omega_1,\omega_2,\hat{\omega})\Big\|^2_{Z_R} \\
&\leqslant \alpha^2_{2m+1} R^{-4m-2} \left\|\mathscr{R}_{d^F,1}(\omega_1,\hat{\omega})\right\|^2_{I_R Y([-\frac{3}{4}R,-\frac{1}{4}R])} \\
&\quad + \alpha^2_{2m+1} R^{-4m-2} \left\|\mathscr{R}_{d^F,2}(\omega_2,\hat{\omega})\right\|^2_{I_R Y([\frac{1}{4}R,\frac{3}{4}R])} \\
&\quad + \alpha^2_{2m+1} R^{-4m-2} \left\|\mathscr{R}_{d^{F,*},1}(\omega_1,\hat{\omega})\right\|^2_{I_R Y([-\frac{3}{4}R,-\frac{1}{4}R])} \\
&\quad + \alpha^2_{2m+1} R^{-4m-2} \left\|\mathscr{R}_{d^{F,*},2}(\omega_2,\hat{\omega})\right\|^2_{I_R Y([\frac{1}{4}R,\frac{3}{4}R])} \ .
\end{aligned} \tag{1.3.53}$$

By Definition 1.2.9, we have

$$\begin{aligned}
&\left\|\mathscr{R}_{d^F,1}(\omega_1,\hat{\omega})\right\|^2_{I_R Y([-\frac{3}{4}R,-\frac{1}{4}R])} + \left\|\mathscr{R}_{d^{F,*},1}(\omega_1,\hat{\omega})\right\|^2_{I_R Y([-\frac{3}{4}R,-\frac{1}{4}R])} \\
&\qquad\qquad \leqslant 2\delta_Y^{-2} e^{-\frac{1}{2}\delta_Y R}\|\omega_1\|^2_{\partial Z_{1,0}} \ , \\
&\left\|\mathscr{R}_{d^F,2}(\omega_2,\hat{\omega})\right\|^2_{I_R Y([\frac{1}{4}R,\frac{3}{4}R])} + \left\|\mathscr{R}_{d^{F,*},2}(\omega_2,\hat{\omega})\right\|^2_{I_R Y([\frac{1}{4}R,\frac{3}{4}R])} \\
&\qquad\qquad \leqslant 2\delta_Y^{-2} e^{-\frac{1}{2}\delta_Y R}\|\omega_2\|^2_{\partial Z_{2,0}} \ .
\end{aligned} \tag{1.3.54}$$

By (1.3.53) and (1.3.54), we have

$$\begin{aligned}
&\left\| D^{F,2m}_{Z_R}\left(F_{Z_R} - G_{Z_R}\right)(\omega_1,\omega_2,\hat{\omega})\right\|^2_{Z_R} \\
&\leqslant \alpha^2_{2m+1}\delta_Y^{-2} R^{-4m-2} e^{-\frac{1}{2}\delta_Y R}\big(\|\omega_1\|^2_{\partial Z_{1,0}} + \|\omega_2\|^2_{\partial Z_{2,0}}\big) \ .
\end{aligned} \tag{1.3.55}$$

Proceeding in the same way as (1.3.43), we have

$$\begin{aligned}
&\|\omega_1\|^2_{\partial Z_{1,0}} + \|\omega_2\|^2_{\partial Z_{2,0}} \\
&\leqslant C\big(\|\omega_1\|^2_{Z_{1,0}} + \|\omega_2\|^2_{Z_{2,0}}\big) = C\big\|(\omega_1,\omega_2,\hat{\omega})\big\|^2_{\mathscr{H}^\bullet(Z_{12,\infty},F)} \ .
\end{aligned} \tag{1.3.56}$$

By (1.3.15), (1.3.55) and (1.3.56), for any $m \in \mathbb{N}$, there exist $c_m > 0$, $R_m > 0$ such that, for any $R > R_m$, any $(\omega_1,\omega_2,\hat{\omega}) \in \mathscr{H}^\bullet(Z_{12,\infty},F)$, we have

$$\Big\| D^{F,2m}_{Z_R}\big(F_{Z_R} - G_{Z_R}\big)(\omega_1,\omega_2,\hat{\omega})\Big\|_{Z_R} \leqslant e^{-c_m R}\|(\omega_1,\omega_2,\hat{\omega})\|_{\mathscr{H}^\bullet(Z_{12,\infty},F)} \ . \tag{1.3.57}$$

Set

$$\begin{aligned}
\mu_0 &= \mathscr{F}_{Z_R}(\omega_1,\omega_2,\hat{\omega}) - \mathscr{G}_{Z_R}(\omega_1,\omega_2,\hat{\omega}) \ , \\
\mu_1 &= F_{Z_R}(\omega_1,\omega_2,\hat{\omega}) - \mathscr{F}_{Z_R}(\omega_1,\omega_2,\hat{\omega}) \ , \\
\mu_2 &= G_{Z_R}(\omega_1,\omega_2,\hat{\omega}) - \mathscr{G}_{Z_R}(\omega_1,\omega_2,\hat{\omega}) \ ,
\end{aligned} \tag{1.3.58}$$

then

$$\big(F_{Z_R} - G_{Z_R}\big)(\omega_1,\omega_2,\hat{\omega}) = \mu_0 + \mu_1 - \mu_2 \ . \tag{1.3.59}$$

By Theorem 1.1.1 and (1.3.30), we have

$$\mu_0 \in \ker\left(D^{F,2}_{Z_R}\right) \ , \quad \mu_1 \in \mathrm{im}(d^F) \ , \quad \mu_2 \in \mathrm{im}(d^{F,*}) \ . \tag{1.3.60}$$

For $m > 0$, by (1.1.6), $D^{F,2m}_{Z_R}$ commutes with d^F and $d^{F,*}$, thus

$$D^{F,2m}_{Z_R}\mu_1 \in \mathrm{im}(d^F) \ , \quad D^{F,2m}_{Z_R}\mu_2 \in \mathrm{im}(d^{F,*}) \ . \tag{1.3.61}$$

As a consequence, $D^{F,2m}_{Z_R}\mu_0$, $D^{F,2m}_{Z_R}\mu_1$ and $D^{F,2m}_{Z_R}\mu_2$ are mutually orhogonal. For $m\in\mathbb{N}$, by (1.3.59) and (1.3.61), we get

$$\begin{aligned}\Big\|D^{F,2m}_{Z_R}(F_{Z_R}-\mathscr{F}_{Z_R})(\omega_1,\omega_2,\hat\omega)\Big\|_{Z_R} &= \Big\|D^{F,2m}_{Z_R}\mu_1\Big\|_{Z_R}\\ &\leqslant \Big\|D^{F,2m}_{Z_R}(F_{Z_R}-G_{Z_R})(\omega_1,\omega_2,\hat\omega)\Big\|_{Z_R}\ .\end{aligned}\tag{1.3.62}$$

By (1.3.57) and (1.3.62), we get

$$\Big\|D^{F,2m}_{Z_R}(F_{Z_R}-\mathscr{F}_{Z_R})(\omega_1,\omega_2,\hat\omega)\Big\|_{Z_R}\leqslant e^{-cR}\big\|(\omega_1,\omega_2,\hat\omega)\big\|_{\mathscr{H}^\bullet(Z_{12,\infty},F)}\ .\tag{1.3.63}$$

By (1.3.63) and Proposition 1.3.5, we get (1.3.49).

The injectivity of $\mathscr{F}_{Z_R}$ follows from (1.3.37) and (1.3.49). □

Remark 1.3.8. The Hodge decomposition is used in an essential way in (1.3.60). The proof of Proposition 1.3.7 cannot be applied to general Dirac operators.

Proposition 1.3.9. *For $\varepsilon>0$, there exists $R_0>0$ such that, for $R>R_0$, any eigensection of $D^F_{Z_R}$ associated with $\lambda\in\,]-R^{-1-\varepsilon},R^{-1-\varepsilon}[$ is contained in the image of $\mathscr{F}_{Z_R}$.*

Proof. Suppose the contrary, i.e., there exist $R_i\to+\infty$, $\omega_i\in\Omega^\bullet(Z_{R_i},F)$ and $\lambda_i\in\,]-R_i^{-1-\varepsilon},R_i^{-1-\varepsilon}[$, such that

$$\omega_i\neq 0\ ,\qquad D^F_{Z_{R_i}}\omega_i=\lambda_i\omega_i\ ,\tag{1.3.64}$$

$$\omega_i\perp \mathrm{im}(\mathscr{F}_{Z_{R_i}})\ .\tag{1.3.65}$$

By Lemma 1.2.1, we may multiply a suitable constant, such that

$$\|\omega_i\|^2_{Z_{R_i}\backslash I_{R_i}Y}=\|\omega_i\|^2_{Z_{1,0}}+\|\omega_i\|^2_{Z_{2,0}}=1\ .\tag{1.3.66}$$

By Lemma 1.2.1 and (1.3.64), there exists $C>0$ such that, for $T\in\mathbb{N}$ and $R_i\geqslant T$,

$$\|\omega_i\|^2_{Z_{1,T}}\leqslant 1+CT\ .\tag{1.3.67}$$

Thus, for any $T\in\mathbb{N}$ fixed, the series $\big(\omega_i|_{Z_{1,T}}\big)_i$ is L^2-bounded.

Since λ_i are bounded, using Rellich's lemma, we may suppose, by extracting a subsequence, that $\big(\omega_i|_{Z_{1,T}}\big)_i$ converges with respect to the k-th Sobolev norm for all $k\in\mathbb{N}$. By the Sobolev imbedding theorem, $\big(\omega_i|_{Z_{1,T}}\big)_i$ converges with respect to the $\mathscr{C}^1$-norm. Using a diagonal argument (involving i and T), we get $\omega_{1,\infty}\in\Omega^\bullet(Z_{1,\infty},F)$ such that, for any $T\in\mathbb{N}$, $\big(\omega_i|_{Z_{1,T}}\big)_i$ converges to $\omega_{1,\infty}|_{Z_{1,T}}$ (with respect to the $\mathscr{C}^1$-norm). Taking the limit of (1.3.64), we get $D^F_{Z_{1,\infty}}\omega_{1,\infty}=0$. Taking the limit of (1.3.67), we get

$$\|\omega_{1,\infty}\|^2_{Z_{1,T}}\leqslant 1+CT\ ,\qquad \text{for } T\in\mathbb{N}\ .\tag{1.3.68}$$

By (1.2.13) and (1.3.68), $\omega_{1,\infty}$ is an extended L^2-solution, i.e., there exists $\hat\omega_1$ such that $(\omega_{1,\infty},\hat\omega_1)\in\mathscr{H}^\bullet(Z_{1,\infty},F)$. In particular,

$$\omega_i^{\mathrm{zm}}\big|_{\partial Z_{1,0}}\to\hat\omega_1\ ,\ \text{as } i\to\infty\ .\tag{1.3.69}$$

Applying the same argument to $\omega_i|_{Z_{2,T}}$, we find $(\omega_{2,\infty},\hat\omega_2)\in\mathscr{H}^\bullet(Z_{2,\infty},F)$ satisfying the same properties. In particular,

$$\omega_i^{\mathrm{zm}}\big|_{\partial Z_{2,0}}\to\hat\omega_2\ ,\ \text{as } i\to\infty\ .\tag{1.3.70}$$

By (1.2.14), we have

$$\omega_i^{\mathrm{zm},\pm}\big|_{\partial Z_{2,0}} = e^{\pm 2\sqrt{-1}R_i\lambda_i}\omega_i^{\mathrm{zm},\pm}\big|_{\partial Z_{1,0}} \ . \tag{1.3.71}$$

Since $R_i\lambda_i \to 0$, by (1.3.69)-(1.3.71), we get

$$\hat{\omega}_1 = \hat{\omega}_2 \ . \tag{1.3.72}$$

Then $(\omega_{1,\infty}, \omega_{2,\infty}, \hat{\omega}_1) \in \mathscr{H}^\bullet(Z_{12,\infty}, F)$.

Set

$$\tilde{\omega}_i = \mathscr{F}_{Z_{R_i}}(\omega_{1,\infty}, \omega_{2,\infty}, \hat{\omega}_1) \ . \tag{1.3.73}$$

Case 1, $\hat{\omega}_1 \neq 0$: We want to show that $\langle \omega_i, \tilde{\omega}_i \rangle \to \infty$ as $i \to \infty$, which contradicts (1.3.65).

We have

$$\langle \omega_i, \tilde{\omega}_i \rangle = \langle \omega_i, \tilde{\omega}_i \rangle_{Z_{R_i}\setminus I_{R_i}Y} + \langle \omega_i^{\mathrm{nz}}, \tilde{\omega}_i^{\mathrm{nz}} \rangle_{I_{R_i}Y} + \langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} \rangle_{I_{R_i}Y} \ . \tag{1.3.74}$$

By Lemma 1.2.1 and Proposition 1.3.7, $\langle \omega_i, \tilde{\omega}_i \rangle_{Z_{R_i}\setminus I_{R_i}Y}$ and $\langle \omega_i^{\mathrm{nz}}, \tilde{\omega}_i^{\mathrm{nz}} \rangle_{I_{R_i}Y}$ are bounded, when $i \to \infty$. It is sufficient to show that $\langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} \rangle_{I_{R_i}Y} \to \infty$ as $i \to \infty$.

We have

$$\langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} \rangle_{I_{R_i}Y} = \langle \omega_i^{\mathrm{zm}}, \pi_Y^*\hat{\omega}_1 \rangle_{I_{R_i}Y} + \langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} - \pi_Y^*\hat{\omega}_1 \rangle_{I_{R_i}Y} \ . \tag{1.3.75}$$

By Definition 1.2.9, 1.3.1, we have

$$\pi_Y^*\hat{\omega}_1 = \Big(F_{Z_{R_i}}(\omega_{1,\infty}, \omega_{2,\infty}, \hat{\omega}_1)\big|_{I_{R_i}Y} \Big)^{\mathrm{zm}} \ . \tag{1.3.76}$$

Then, by Proposition 1.3.7,

$$\langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} - \pi_Y^*\hat{\omega}_1 \rangle_{I_{R_i}Y} \to 0, \ \text{as } i \to \infty \ . \tag{1.3.77}$$

By (1.2.14) and the fact that $R_i\lambda_i \to 0$, the restriction of ω_i^{zm} to $I_{R_i}Y(u)$ ($u \in [-R_i, R_i]$) converges uniformly to the same limit. Then, by (1.3.69), they all converge to $\hat{\omega}_1$. Thus,

$$\langle \omega_i^{\mathrm{zm}}, \pi_Y^*\hat{\omega}_1 \rangle_{I_{R_i}Y} = \int_{-R_i}^{R_i} \langle \omega_i^{\mathrm{zm}}|_{I_{R_i}Y(u)}, \hat{\omega}_1 \rangle_Y du \to +\infty \ , \ \text{as } i \to \infty \ . \tag{1.3.78}$$

This ends the first case.

Case 2, $\hat{\omega}_1 = 0$: We want to show that

$$\langle \omega_i, \tilde{\omega}_i \rangle \to \|\omega_{1,\infty}\|^2_{Z_{1,\infty}} + \|\omega_{2,\infty}\|^2_{Z_{2,\infty}} > 0 \ , \ \text{as } i \to \infty \ , \tag{1.3.79}$$

which contradicts (1.3.65).

For any $T > 0$, $R_i > T$, we have

$$\langle \omega_i, \tilde{\omega}_i \rangle = \langle \omega_i, \tilde{\omega}_i \rangle_{Z_{1,T}\cup Z_{2,T}} + \langle \omega_i^{\mathrm{nz}}, \tilde{\omega}_i^{\mathrm{nz}} \rangle_{I_{R_i}Y([-R_i+T, R_i-T])} + \langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} \rangle_{I_{R_i}Y([-R_i+T, R_i-T])} \ . \tag{1.3.80}$$

By Definition 1.3.1 and Proposition 1.3.7,

$$\langle \omega_i, \tilde{\omega}_i \rangle_{Z_{1,T}\cup Z_{2,T}} \to \|\omega_{1,\infty}\|^2_{Z_{1,T}} + \|\omega_{2,\infty}\|^2_{Z_{2,T}} \ , \ \text{as } i \to \infty \ . \tag{1.3.81}$$

By Lemma 1.2.1, if $R_i > \delta_Y^{-1}$ and $\lambda_i < \frac{1}{2}\delta_Y$ (which hold for i large enough),

$$\begin{aligned} &\big| \langle \omega_i^{\mathrm{nz}}, \tilde{\omega}_i^{\mathrm{nz}} \rangle_{I_{R_i}Y([-R_i+T, R_i-T])} \big| \\ &\leqslant \ 8\delta_Y^{-1}(\|\omega_i\|_{\partial Z_{1,T}} + \|\omega_i\|_{\partial Z_{2,T}})(\|\tilde{\omega}_i\|_{\partial Z_{1,T}} + \|\tilde{\omega}_i\|_{\partial Z_{2,T}}) \ . \end{aligned} \tag{1.3.82}$$

Furthermore, as $i \to \infty$,

$$(\|\omega_i\|_{\partial Z_{1,T}} + \|\omega_i\|_{\partial Z_{2,T}})(\|\tilde{\omega}_i\|_{\partial Z_{1,T}} + \|\tilde{\omega}_i\|_{\partial Z_{2,T}}) \to (\|\omega_{1,\infty}\|_{\partial Z_{1,T}} + \|\omega_{2,\infty}\|_{\partial Z_{2,T}})^2 , \tag{1.3.83}$$

and

$$(\|\omega_{1,\infty}\|_{\partial Z_{1,T}} + \|\omega_{2,\infty}\|_{\partial Z_{2,T}})^2 \leqslant e^{-2\delta_Y T}(\|\omega_{1,\infty}\|_{\partial Z_{1,0}} + \|\omega_{2,\infty}\|_{\partial Z_{2,0}})^2 . \tag{1.3.84}$$

Since $\hat{\omega}_1 = 0$, proceeding in same way as (1.3.77), we get

$$\langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} \rangle_{I_{R_i}Y([-R_i+T,R_i-T])} \to 0, \text{ as } i \to \infty . \tag{1.3.85}$$

By (1.3.80)-(1.3.85), we get

$$\limsup_{i\to\infty} \left| \langle \omega_i, \tilde{\omega}_i \rangle - \|\omega_{1,\infty}\|^2_{Z_{1,T}} - \|\omega_{2,\infty}\|^2_{Z_{2,T}} \right| \leqslant 8\delta_Y^{-1} e^{-2\delta_Y T}(\|\omega_{1,\infty}\|_{\partial Z_{1,0}} + \|\omega_{2,\infty}\|_{\partial Z_{2,0}})^2 . \tag{1.3.86}$$

Taking $T \to \infty$ in (1.3.86), we get (1.3.79). □

Theorem 1.3.10. *There exists $R_0 > 0$ such that, for $R > R_0$, the map $\mathscr{F}_{Z_R} : \mathscr{H}^\bullet(Z_{12,\infty}, F) \to \ker(D^{F,2}_{Z_R})$ is bijective. Moreover,*

$$\mathrm{Sp}(D^F_{Z_R}) \subseteq]-\infty, -R^{-1-\varepsilon}[\cup \{0\} \cup]R^{-1-\varepsilon}, +\infty[. \tag{1.3.87}$$

Proof. These are direct consequences of Proposition 1.3.7, Proposition 1.3.9. □

1.3.4. *Approximating the small eigenvalues.*

For $j = 1, 2$, let $D^F_{Z_{j,\infty},\mathrm{pp}}$ be the restriction of $D^F_{Z_{j,\infty}}$ to its p.p. spectrum (cf. §1.2.3). We fix $\delta_{Z_j} > 0$ such that $\mathrm{Sp}\left(D^F_{Z_{j,\infty},\mathrm{pp}}\right) \cap [-\delta_{Z_j}, \delta_{Z_j}] \subseteq \{0\}$. Put $\delta = \frac{1}{2}\min\{\delta_Y, \delta_{Z_1}, \delta_{Z_2}\}$.

We recall that $\mathscr{E}_{A,R}(Z_{12,\infty}, F)$ was defined in (1.3.21), $I_{1,R}Y, I_{2,R}Y, I_RY$ were defined at the end of §1.3.1 and $\chi^\pm_R$ was defined at the beginning of §1.3.3.

Definition 1.3.11. We define

$$J_{A,Z_R} : \mathscr{E}_{A,R}(Z_{12,\infty}, F) \to \Omega^\bullet(Z_R, F) , \tag{1.3.88}$$

such that for any $(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) \in \mathscr{E}_{A,R}(Z_{12,\infty}, F)$,

$$\begin{aligned} & J_{A,Z_R}(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})\big|_{Z_{j,0}} = \omega_j , \qquad \text{for } j = 1, 2 , \\ & J_{A,Z_R}(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})\big|_{I_RY} = \chi_{1,R}\, \omega_1\big|_{I_{1,2R}Y} + \chi_{2,R}\, \omega_2\big|_{I_{2,2R}Y} \\ & \qquad\qquad + \big(1 - \chi_{1,R} - \chi_{2,R}\big)\omega_1^{\mathrm{zm}}\big|_{I_{1,2R}Y} . \end{aligned} \tag{1.3.89}$$

Here we identify $I_{j,2R}Y$ $(j = 1, 2)$ to I_RY. Then $\omega_j|_{I_{j,2R}Y}$ $(j = 1, 2)$ and $\omega_1^{\mathrm{zm}}\big|_{I_{1,2R}Y}$ are viewed as sections on I_RY.

Let $\mathscr{E}_B(Z_R, F) \subseteq \Omega^\bullet(Z_R, F)$ be the eigenspace of $D^F_{Z_R}$ associated with the eigenvalues in B. Let $P^B_{Z_R} : \Omega^\bullet(Z_R, F) \to \mathscr{E}_B(Z_R, F)$ be the orthogonal projection.

Definition 1.3.12. Set

$$\mathscr{J}_{A,B,Z_R} = P^B_{Z_R} \circ J_{A,Z_R} : \mathscr{E}_{A,R}(Z_{12,\infty}, F) \to \mathscr{E}_B(Z_R, F) . \tag{1.3.90}$$

For $A, B \subseteq \mathbb{R}$ and $\alpha > 0$, we denote $A \subseteq_\alpha B$, if $]x - \alpha, x + \alpha[\subseteq B$ for any $x \in A$.

Proposition 1.3.13. *There exist $R_0 > 0$, $c > 0$ such that for $R > R_0$, $A \subseteq_{e^{-cR}} B \subseteq]-\delta, 0[\cup]0, \delta[$ and $(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) \in \mathscr{E}_{A,R}(Z_{12,\infty}, F)$, we have*

$$\begin{aligned} &\big\|\big(\mathscr{J}_{A,B,Z_R} - J_{A,Z_R}\big)(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})\big\|_{\mathscr{C}^0,Z_R} \\ &\leqslant e^{-cR}\big\|(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})\big\|_{\mathscr{E}_{A,R}(Z_{12,\infty},F)} . \end{aligned} \tag{1.3.91}$$

As a consequence, $\mathscr{J}_{A,B,Z_R}$ is injective for R large enough.

Proof. It suffices to consider the case $(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) \in \mathscr{E}^\bullet_{\lambda_0,R}(Z_{12,\infty}, F)$ with $\lambda_0 \in A$.

Proceeding in the same way as (1.3.57), for any $m \in \mathbb{N}$, there exist $R_m > 0$, $c_m > 0$ such that for $R > R_m$,

$$\begin{aligned} &\Big\|D^{F,m}_{Z_R}(D^F_{Z_R} - \lambda_0)J_{A,Z_R}(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})\Big\|^2_{Z_R} \\ &\leqslant e^{-3c_mR}\,\|(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})\|^2_{\mathscr{E}_{A,R}(Z_{12,\infty},F)} . \end{aligned} \tag{1.3.92}$$

We have the decomposition

$$J_{A,Z_R}(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) = \sum_\lambda \mu_\lambda \tag{1.3.93}$$

with $D^F_{Z_R}\mu_\lambda = \lambda\mu_\lambda$. In particuler, these μ_λ are mutually orthogonal. Then

$$\mathscr{J}_{A,B,Z_R}(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) = \sum_{\lambda\in B} \mu_\lambda . \tag{1.3.94}$$

By (1.3.92) and (1.3.93), we have

$$\sum_{|\lambda-\lambda_0|>e^{-c_mR}} \Big\|D^{F,m}_{Z_R}\mu_\lambda\Big\|^2_{Z_R} \leqslant e^{-2c_mR}\big\|(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})\big\|^2_{\mathscr{E}_{A,R}(Z_{12,\infty},F)} . \tag{1.3.95}$$

By Proposition 1.3.5 and (1.3.93)-(1.3.95), we get (1.3.91). □

Lemma 1.3.14. *For $\varepsilon > 0$, there exist $R_0 > 0$, $C > 0$ such that for any $R > R_0$, $\omega \in \Omega^\bullet(Z_R, F)$ an eigensection associated with $\lambda \in]-\delta+\varepsilon, 0[\cup]0, \delta-\varepsilon[$, we have*

$$\|\omega^{\mathrm{zm},+}\|^2_Y + \|\omega^{\mathrm{zm},-}\|^2_Y \geqslant C\|\omega\|^2_{Z_{1,0}\cup Z_{2,0}} . \tag{1.3.96}$$

In particular, ω^{zm} is non zero.

Proof. Suppose the contrary, i.e., there exist $R_i \to +\infty$, $\omega_i \in \Omega^\bullet(Z_{R_i}, F)$ and $\lambda_i \in]-\delta+\varepsilon, 0[\cup]0, \delta-\varepsilon[$, such that

$$D^F_{Z_{R_i}}\omega_i = \lambda_i\omega_i , \tag{1.3.97}$$

and

$$\frac{\|\omega_i^{\mathrm{zm},+}\|^2_Y + \|\omega_i^{\mathrm{zm},-}\|^2_Y}{\|\omega_i\|^2_{Z_{1,0}\cup Z_{2,0}}} \to 0 , \quad \text{as } i\to\infty . \tag{1.3.98}$$

By extracting a subsequence, we may assume that $\lambda_i \to \lambda_\infty$. By Lemma 1.2.1, $\|\omega_i\|^2_{Z_{1,0}\cup Z_{2,0}} \neq 0$, we may multiply suitable constants such that

$$\|\omega_i\|^2_{Z_{1,0}\cup Z_{2,0}} = 1 . \tag{1.3.99}$$

By (1.3.98) and (1.3.99), we have

$$\|\omega_i^{\mathrm{zm},+}\|^2_Y + \|\omega_i^{\mathrm{zm},-}\|^2_Y \to 0 , \quad \text{as } i\to\infty. \tag{1.3.100}$$

Proceeding in the same way as in the proof of Proposition 1.3.9, by extracting a subsequence, we may assume that there exist $\omega_{1,\infty} \in \Omega^\bullet(Z_{1,\infty}, F)$, $\omega_{2,\infty} \in \Omega^\bullet(Z_{2,\infty}, F)$ such that for any $T \in \mathbb{N}$, $\left(\omega_i\big|_{Z_{j,T}}\right)_i$ converges to $\omega_{j,\infty}\big|_{Z_{j,T}}$ $(j = 1, 2)$ with respect to the $\mathscr{C}^1$-norm. Taking the limit of (1.3.97) and (1.3.100), we get

$$D^F_{Z_{j,\infty}} \omega_{j,\infty} = \lambda_\infty \omega_{j,\infty}\ , \quad \omega^{\mathrm{zm},\pm}_{j,\infty} = 0\ , \qquad \text{for } j = 1, 2\ . \tag{1.3.101}$$

Taking the limit of (1.3.99), we get

$$\|\omega_{1,\infty}\|^2_{Z_{1,0}} + \|\omega_{2,\infty}\|^2_{Z_{2,0}} = 1\ . \tag{1.3.102}$$

Thus one of $\omega_{1,\infty}$, $\omega_{2,\infty}$ is non zero. We may assume that $\omega_{1,\infty}$ is non zero. By (1.3.101), $\omega_{1,\infty}$ is zeromode free, thus, a L^2-eigensection (by Lemma 1.2.1). We get $\lambda_\infty \in \mathrm{Sp}\left(D^F_{Z_{1,\infty},\mathrm{pp}}\right)$. But $|\lambda_\infty| < \delta \leqslant \delta_{Z_1}$, by the definition of δ_{Z_1}, we must have $\lambda_\infty = 0$. Thus $\omega_{1,\infty} \in \mathscr{H}^\bullet_{L^2}(Z_{1,\infty}, F)$.

Recall that $\mathscr{F}_{Z_{R_i}}(\cdot,\cdot,\cdot)$ was defined in (1.3.48). Proceeding in the same way as (1.3.79), we get

$$\left\langle \omega_i, \mathscr{F}_{Z_{R_i}}(\omega_{1,\infty}, 0, 0)\right\rangle \to \|\omega_{1,\infty}\|^2 > 0\ , \quad \text{as } i \to \infty. \tag{1.3.103}$$

But, by (1.3.97), $\lambda_i \neq 0$ and $\mathscr{F}_{Z_{R_i}}(\omega_{1,\infty}, 0, 0) \in \ker\left(D^{F,2}_{Z_{R_i}}\right)$, we have $\omega_i \perp \mathscr{F}_{Z_{R_i}}(\omega_{1,\infty}, 0, 0)$. This contradicts (1.3.103). □

Lemma 1.3.15. *For $\varepsilon > 0$, there exist $R_0 > 0$, $c > 0$ such that for $R > R_0$ and $\omega \in \Omega^\bullet(Z_R, F)$ an eigensection of $D^F_{Z_R}$ associated with $\lambda \in]-\delta+\varepsilon, 0[\cup]0, \delta-\varepsilon[$, we have*

$$\left\|C_j(\lambda)\omega^{\mathrm{zm},-}\big|_{\partial Z_{j,0}} - \omega^{\mathrm{zm},+}\big|_{\partial Z_{j,0}}\right\|_Y \leqslant e^{-cR}\|\omega\|_{Z_{1,0}\cup Z_{2,0}}\ , \qquad \textit{for } j = 1, 2\ . \tag{1.3.104}$$

In particular,

$$\left\|\left(e^{4i\lambda R} C_{12}(\lambda) - 1\right)\omega^{\mathrm{zm},-}\big|_{\partial Z_{1,0}}\right\|_Y \leqslant e^{-cR}\|\omega\|_{Z_{1,0}\cup Z_{2,0}}\ . \tag{1.3.105}$$

Proof. We follow the argument in [M94, PW06].

We will only establish (1.3.104) for $j = 1$.

Let $\omega \in \Omega^\bullet(Z_R, F)$ be an eigensection of $D^F_{Z_R}$ associated with $\lambda \in]-\delta+\varepsilon, 0[\cup]0, \delta-\varepsilon[$. By (1.2.13), there exist $\phi, \phi' \in \mathscr{H}^\bullet(Y, F)$ such that

$$\omega|_{I_{1,R}Y} = e^{-i\lambda u_1}(\phi - ic(\tfrac{\partial}{\partial u})\phi) + e^{i\lambda u_1}(\phi' + ic(\tfrac{\partial}{\partial u})\phi') + \omega^{\mathrm{nz}}\ . \tag{1.3.106}$$

By Proposition 1.2.4, there exists $(\tilde{\omega}, \tilde{\omega}^{\mathrm{zm}}) \in \mathscr{E}^\bullet_\lambda(Z_{1,\infty}, F)$ satisfying

$$\tilde{\omega}^{\mathrm{zm}} = e^{-i\lambda u_1}(\phi - ic(\tfrac{\partial}{\partial u})\phi) + e^{i\lambda u_1} C_1(\lambda)(\phi - ic(\tfrac{\partial}{\partial u})\phi)\ . \tag{1.3.107}$$

Set

$$\mu = \omega - \tilde{\omega} \in \Omega^\bullet(Z_{1,R}, F)\ . \tag{1.3.108}$$

Then μ is also an eigensection of $D^F_{Z_R}$ associated with λ. Thus

$$\langle D^F_{Z_R}\mu, \mu\rangle_{Z_{1,R}} - \langle \mu, D^F_{Z_R}\mu\rangle_{Z_{1,R}} = \langle \lambda\mu, \mu\rangle_{Z_{1,R}} - \langle \mu, \lambda\mu\rangle_{Z_{1,R}} = 0\ . \tag{1.3.109}$$

On the other hand, by (1.2.11) and (1.3.106)-(1.3.108), we have

$$\begin{aligned} &\langle D^F_{Z_R}\mu, \mu\rangle_{Z_{1,R}} - \langle \mu, D^F_{Z_R}\mu\rangle_{Z_{1,R}} \\ &= \langle c(\tfrac{\partial}{\partial u})\mu, \mu\rangle_{\partial Z_{1,R}} = -2i\ \|\phi' - C_1(\lambda)\phi\|^2_Y + \langle c(\tfrac{\partial}{\partial u})\mu^{\mathrm{nz}}, \mu^{\mathrm{nz}}\rangle_{\partial Z_{1,R}}\ . \end{aligned} \tag{1.3.110}$$

By (1.3.106) and (1.3.108)-(1.3.110), we get

$$\begin{aligned}&\left\|C_1(\lambda)\omega^{\mathrm{zm},-}|_{\partial Z_{1,0}}-\omega^{\mathrm{zm},+}|_{\partial Z_{1,0}}\right\|_Y^2\\ &= -i\langle c(\tfrac{\partial}{\partial u})\mu^{\mathrm{nz}},\mu^{\mathrm{nz}}\rangle_{\partial Z_{1,R}}\leqslant\|\mu^{\mathrm{nz}}\|^2_{\partial Z_{1,R}}\leqslant\|\omega^{\mathrm{nz}}\|^2_{\partial Z_{1,R}}+\|\tilde\omega^{\mathrm{nz}}\|^2_{\partial Z_{1,R}}\ .\end{aligned}\tag{1.3.111}$$

By (1.2.18), we have

$$\|\omega^{\mathrm{nz}}\|^2_{\partial Z_{1,R}}\leqslant e^{-\varepsilon R}\|\omega\|^2_{\partial Z_{1,0}\,\cup\,\partial Z_{2,0}}\ ,\quad \|\tilde\omega^{\mathrm{nz}}\|^2_{\partial Z_{1,R}}\leqslant e^{-\varepsilon R}\|\tilde\omega\|^2_{\partial Z_{1,0}}\ .\tag{1.3.112}$$

By (1.3.56), there exists $C_1>0$ depending on Z_1,Z_2,F such that

$$\|\omega\|^2_{\partial Z_{1,0}\,\cup\,\partial Z_{2,0}}\leqslant C_1\|\omega\|^2_{Z_{1,0}\,\cup\,Z_{2,0}}\ ,\quad \|\tilde\omega\|^2_{\partial Z_{1,0}}\leqslant C_1\|\tilde\omega\|^2_{Z_{1,0}}\ .\tag{1.3.113}$$

By (1.2.33) and (1.3.106), we have

$$\|\tilde\omega\|^2_{Z_{1,0}}\leqslant C_2\|\phi-ic(\tfrac{\partial}{\partial u})\phi\|^2_Y\leqslant C_2\|\omega\|^2_{\partial Z_{1,0}}\ .\tag{1.3.114}$$

Combining (1.3.111)-(1.3.114), the proof is terminated. □

Lemma 1.3.16. *For $\varepsilon>0$, there exist $R_0>0$, $c>0$ such that for any $R>R_0$, $\omega\in\Omega^\bullet(Z_R,F)$ an eigensection associated with $\lambda_0\in]-\delta+\varepsilon,0[\cup]0,\delta-\varepsilon[$, there exists $\tilde\omega\in\mathrm{im}\left(\mathscr{J}_{]\lambda_0-e^{-cR},\lambda_0+e^{-cR}[,]-\delta,0[\cup]0,\delta[,Z_R}\right)$ satisfying*

$$\|\omega^{\mathrm{zm}}-\tilde\omega^{\mathrm{zm}}\|_{I_RY}\leqslant e^{-cR}\|\omega\|_{Z_{1,0}\,\cup\,Z_{2,0}}\ .\tag{1.3.115}$$

Proof. We claim that there exist $c>0$, $C>0$, $R_0>0$ such that for any $R>R_0$, $\omega\in\Omega^\bullet(Z_R,F)$ an eigensection associated with $\lambda_0\in]-\delta+\varepsilon,0[\,\cup\,]0,\delta-\varepsilon[$, there exists $\mu\in\mathrm{im}(J_{]\lambda_0-e^{-cR},\lambda_0+e^{-cR}[,Z_R})$, such that

$$\|\omega^{\mathrm{zm}}-\mu^{\mathrm{zm}}\|_{I_RY}\leqslant e^{-cR}\|\omega\|_{Z_{1,0}\,\cup\,Z_{2,0}}\ ,\qquad \|\mu\|_{Z_{1,0}\cup Z_{2,0}}\leqslant C\|\omega\|_{Z_{1,0}\,\cup\,Z_{2,0}}\ .\tag{1.3.116}$$

Once (1.3.116) is proved, (1.3.115) follows : for R large enough, by Theorem 1.3.10, we have

$$]\lambda_0-e^{-cR},\lambda_0+e^{-cR}[\ \subseteq_{e^{-cR}}]-\delta,0[\cup]0,\delta[\ .\tag{1.3.117}$$

Let $(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}})\in\mathscr{E}_{]\lambda_0-e^{-cR},\lambda_0+e^{-cR}[,R}(Z_{12,\infty},F)$, such that

$$\mu=J_{]\lambda_0-e^{-cR},\lambda_0+e^{-cR}[,Z_R}(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}})\ .\tag{1.3.118}$$

By Definition 1.3.11, we have

$$\|\mu\|_{Z_{1,0}\cup Z_{2,0}}=\left\|(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}})\right\|_{\mathscr{E}_{]\lambda_0-e^{-cR},\lambda_0+e^{-cR}[,R}(Z_{12,\infty},F)}\ .\tag{1.3.119}$$

Set

$$\tilde\omega=\mathscr{J}_{]\lambda_0-e^{-cR},\lambda_0+e^{-cR}[,]-\delta,0[\cup]0,\delta[,Z_R}(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}})\ .\tag{1.3.120}$$

by Proposition 1.3.13, (1.3.116), (1.3.117) and (1.3.119), we get (1.3.115).

Now we prove (1.3.116).

Since ω is an eigensection of $D^F_{Z_R}$ associated with λ_0, we have

$$\omega^{\mathrm{zm}}=e^{-i\lambda_0u_1}(\omega^{\mathrm{zm},-}|_{\partial Z_{1,0}})+e^{i\lambda_0u_1}(\omega^{\mathrm{zm},+}|_{\partial Z_{1,0}})\ .\tag{1.3.121}$$

By Lemma 1.3.15, we have

$$\begin{aligned}&\left\|C_1(\lambda_0)\omega^{\mathrm{zm},-}|_{\partial Z_{1,0}}-\omega^{\mathrm{zm},+}|_{\partial Z_{1,0}}\right\|_Y\leqslant e^{-cR}\|\omega\|_{Z_{1,0}\cup Z_{2,0}}\ ,\\ &\left\|e^{4i\lambda_0R}C_{12}(\lambda_0)\omega^{\mathrm{zm},-}|_{\partial Z_{1,0}}-\omega^{\mathrm{zm},-}|_{\partial Z_{1,0}}\right\|_Y\leqslant e^{-cR}\|\omega\|_{Z_{1,0}\cup Z_{2,0}}\ .\end{aligned}\tag{1.3.122}$$

We proceed in the same way as (1.3.56): using Trace Theorem and elliptic estimate, we get

$$\|\omega^{\mathrm{zm},-}|_{\partial Z_{1,0}}\|_Y \leqslant \|\omega|_{\partial Z_{1,0}}\|_Y \leqslant C\|\omega\|_{Z_{1,0}\cup Z_{2,0}} \ . \tag{1.3.123}$$

By (1.3.96) and (1.3.122), we have

$$\|\omega\|_{Z_{1,0}\cup Z_{2,0}} \leqslant C\|\omega^{\mathrm{zm},-}|_{\partial Z_{1,0}}\|_Y \ . \tag{1.3.124}$$

By Proposition 1.8.2, (1.3.122), (1.3.123) and (1.3.124), there exist $\phi_j \in \mathscr{H}^\bullet(Y,F[du])$, $\lambda_j \in \mathbb{R}$ and $\varphi_j \in \mathscr{H}^\bullet(Y,F[du])$ $(j = 1,\cdots,\dim\mathscr{H}^\bullet(Y,F[du]))$ such that the following orthogonal decomposition holds,

$$\omega^{\mathrm{zm},-}\big|_{\partial Z_{1,0}} = \sum_{j=1}^{\dim\mathscr{H}^\bullet(Y,F[du])} \phi_j \ , \tag{1.3.125}$$

and

$$\begin{aligned} &|\lambda_j - \lambda_0| < e^{-cR} \ , \qquad \|\varphi_j - \phi_j\|_Y < e^{-cR}\|\omega\|_{Z_{1,0}\cup Z_{2,0}} \ , \\ &e^{4iR\lambda_j}C_{12}(\lambda_j)\varphi_j = \varphi_j \ . \end{aligned} \tag{1.3.126}$$

By (1.3.17) and (1.3.21), we can find $(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}}) \in \mathscr{E}_{A,R}(Z_{12,\infty},F)$ satisfying

$$\omega_1^{\mathrm{zm}} = \sum_{j=1}^{\dim\mathscr{H}^\bullet(Y,F[du])} \left(e^{-i\lambda_j u_1}\varphi_j + e^{i\lambda_j u_1}C_1(\lambda_j)\varphi_j\right) \ . \tag{1.3.127}$$

Put

$$\mu = J_{A,Z_R}(\omega_1,\omega_1^{\mathrm{zm}},\omega_2,\omega_2^{\mathrm{zm}}) \ . \tag{1.3.128}$$

Then, under the identification $I_RY \simeq I_{1,2R}Y \subseteq I_{1,\infty}Y$, we have

$$\mu^{\mathrm{zm}} = \omega_1^{\mathrm{zm}} \ . \tag{1.3.129}$$

We prove the first inequality in (1.3.116). By (1.3.121), (1.3.127) and (1.3.129), it suffices to show that, for $u_1 \in [0,2R]$,

$$\begin{aligned} \Bigg\|e^{-i\lambda_0 u_1}\left(\omega^{\mathrm{zm},-}\big|_{\partial Z_{1,0}}\right) + e^{i\lambda_0 u_1}\left(\omega^{\mathrm{zm},+}\big|_{\partial Z_{1,0}}\right) \\ - \sum_{j=1}^{\dim\mathscr{H}^\bullet(Y,F[du])} \left(e^{-i\lambda_j u_1}\varphi_j + e^{i\lambda_j u_1}C_1(\lambda_j)\varphi_j\right)\Bigg\|_Y \leqslant e^{-cR}\|\omega\|_{Z_{1,0}\cup Z_{2,0}} \ , \end{aligned} \tag{1.3.130}$$

which is a consequence of (1.3.122), (1.3.125) and (1.3.126).

We prove the second inequality in (1.3.116). By Definition 1.3.11, (1.2.33), (1.3.127) and (1.3.128), it suffices to show that

$$\sum_{j=1}^{\dim\mathscr{H}^\bullet(Y,F[du])} \|\varphi_j\|_Y \leqslant C\|\omega\|_{Z_{1,0}\,\cup\, Z_{2,0}} \ , \tag{1.3.131}$$

which follows from (1.3.123), (1.3.125) and (1.3.126). □

Proposition 1.3.17. *For $\varepsilon > 0$, there exist $R_0 > 0$, $c > 0$ such that for $R > R_0$ and $B \subseteq_{e^{-cR}} A \subseteq]-\delta+\varepsilon,0[\,\cup\,]0,\delta-\varepsilon[$, $\mathscr{J}_{A,B,Z_R}$ is surjective.*

Proof. Suppose the contrary, i.e., there exist $R_i \to +\infty$, $\omega_i \in \Omega^\bullet(Z_{R_i}, F)$ and $\lambda_i \in B$ satisfying

$$D^F_{Z_{R_i}} \omega_i = \lambda_i \omega_i \ , \quad \omega_i \perp \mathrm{im}(\mathscr{J}_{A,B,Z_{R_i}}) \ . \tag{1.3.132}$$

By Definition 1.3.12, we have

$$\mathrm{im}\left(\mathscr{J}_{A,]-\delta,0[\,\cup\,]0,\delta[,Z_{R_i}}\right) = \mathrm{im}\left(\mathscr{J}_{A,B,Z_{R_i}}\right) \oplus \mathrm{im}\left(\mathscr{J}_{A,]-\delta,0[\,\cup\,]0,\delta[\backslash B,Z_{R_i}}\right) \ . \tag{1.3.133}$$

Furthermore, $\mathscr{J}_{A,]-\delta,0[\,\cup\,]0,\delta[\backslash B,Z_{R_i}}$ is spanned by the eigensections associated with those $\lambda \in]-\delta,0[\,\cup\,]0,\delta[\backslash B$. By (1.3.132), we have

$$\omega_i \perp \mathrm{im}(\mathscr{J}_{A,]-\delta,0[\,\cup\,]0,\delta[,Z_{R_i}}) \ . \tag{1.3.134}$$

By multiplying suitable constants, we may assume that

$$\|\omega_i\|_{Z_{1,0}\,\cup\, Z_{2,0}} = 1 \ . \tag{1.3.135}$$

Then, by Proposition 1.3.14, we have

$$\|\omega_i^{\mathrm{zm},+}\|^2_Y + \|\omega_i^{\mathrm{zm},-}\|^2_Y \geqslant c > 0 \ . \tag{1.3.136}$$

By Lemma 1.3.16, there exists $\tilde{\omega}_i \in \mathrm{im}(\mathscr{J}_{A,]-\delta,0[\,\cup\,]0,\delta[,Z_{R_i}})$ such that

$$\|\omega_i^{\mathrm{zm}} - \tilde{\omega}_i^{\mathrm{zm}}\|_{I_{R_i}Y} \to 0 \ , \ \text{as } i \to \infty \ . \tag{1.3.137}$$

By (1.3.136), (1.3.137), we have

$$\langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} \rangle \to \infty \ , \ \text{as } i \to \infty \ . \tag{1.3.138}$$

By Lemma 1.2.1 and (1.3.135), there exists $C > 0$, such that

$$\left| \langle \omega_i, \tilde{\omega}_i \rangle - \langle \omega_i^{\mathrm{zm}}, \tilde{\omega}_i^{\mathrm{zm}} \rangle \right| \leqslant C \ . \tag{1.3.139}$$

Then, by (1.3.138), $\langle \omega_i, \tilde{\omega}_i \rangle$ tends to ∞. This contradicts (1.3.132). □

Theorem 1.3.18. *For $\varepsilon > 0$, there exists $R_0 > 0$ such that for $R > R_0$, we have*

$$\mathrm{Sp}\left(D^F_{Z_R}\right) \subseteq \ \left]-\infty, -R^{-1-\varepsilon}\right[\ \cup \ \{0\} \ \cup \ \left]R^{-1-\varepsilon}, \infty\right[\ . \tag{1.3.140}$$

Furthermore, if we denote

(1.3.141)

$$\Lambda_R \backslash \{0\} = \left\{ \lambda_k \ : \ k \in \mathbb{Z}\backslash\{0\} \right\} , \quad \textit{with } \cdots \leqslant \lambda_{-1} < 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \ ,$$

$$\mathrm{Sp}\left(D^F_{Z_R}\right) \backslash \{0\} = \left\{ \rho_k \ : \ k \in \mathbb{Z}\backslash\{0\} \right\} , \quad \textit{with } \cdots \leqslant \rho_{-1} < 0 < \rho_1 \leqslant \rho_2 \leqslant \cdots \ ,$$

there exist $\gamma, c > 0$ such that for $R > R_0$ and $|\lambda_k| < \gamma$,

$$|\lambda_k - \rho_k| \leqslant e^{-cR} \ . \tag{1.3.142}$$

Proof. The first part is equivalent to Theorem 1.3.10. We prove the second part.

We fix ε, c and R_0 such that Theorem 1.3.10, Proposition 1.3.13 and Proposition 1.3.17 hold. We enlarge R_0 such that, for $R > R_0$,

$$\varepsilon > R^{-1-\varepsilon} > e^{-cR} \tag{1.3.143}$$

By Theorem 1.8.1, we have

$$\Lambda_R = \bigcup_{k=1}^m \left\{ \lambda \in \mathbb{R} \ : \ 4R\lambda + \theta_k(\lambda) \in 2\pi\mathbb{Z} \right\} , \tag{1.3.144}$$

where $\theta_1(\lambda), \cdots, \theta_m(\lambda)$ are analytic functions on λ such that $\left\{e^{i\theta_1(\lambda)}, \cdots, e^{i\theta_m(\lambda)}\right\} = \mathrm{Sp}\big(C_{12}(\lambda)\big)$. By enlarging R_0, we can show that for $R > R_0$,

$$\Lambda_R \subseteq \,]-\infty, -R^{-1-\varepsilon}[\, \cup \{0\} \cup \,]R^{-1-\varepsilon}, \infty[\,. \tag{1.3.145}$$

For $k > 0$, if $\lambda_k < \delta - \varepsilon$, we apply Proposition 1.3.17 with

$$A =]0, \lambda_k[\,, \qquad B =]R^{-1-\varepsilon}, \lambda_k - e^{-cR}[\,. \tag{1.3.146}$$

(By (1.3.143) and (1.3.145), we have $B \subseteq_{e^{-cR}} A$.) Then $\mathscr{J}_{A,B,Z_R}$ is surjective. As a consequence, $D^F_{Z_R}$ has at most $k-1$ eigenvalues lying in B. Further, by Theorem 1.3.10, we have $\rho_1 > R^{-1-\varepsilon}$. Then we must have $\rho_k \geqslant \lambda_k - e^{-cR}$. A similar argument using Proposition 1.3.13 shows that $\rho_k \leqslant \lambda_k + e^{-cR}$. For $k < 0$, we have parallel arguments.

Set $\gamma = \delta - \varepsilon$, then (1.3.142) holds. □

For $0 \leqslant p \leqslant \dim Z$, we set

$$\begin{aligned} &C^p_{12}(\lambda) = C_{12}(\lambda)\big|_{\mathscr{H}^p(Y,F)\oplus\mathscr{H}^{p-1}(Y,F)du}\,, \\ &\Lambda^p_R = \Big\{\lambda > 0 \ : \ \det\big(e^{4i\lambda R}C^p_{12}(\lambda) - 1\big) = 0\Big\}\,. \end{aligned} \tag{1.3.147}$$

Let $D^{F,2,(p)}_{Z_R}$ be the restriction of $D^{F,2}_{Z_R}$ on $\Omega^p(Z_R, F)$.

Theorem 1.3.19. *If we denote*

$$\begin{aligned} \Lambda^p_R &= \Big\{\lambda_k \ : \ k = 1, 2, \cdots\Big\}\,, \qquad \text{with } 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots\,, \\ \mathrm{Sp}\left(D^{F,2,(p)}_{Z_R}\right)\backslash\{0\} &= \Big\{\rho_k \ : \ k = 1, 2, \cdots\Big\}\,, \qquad \text{with } 0 < \rho_1 \leqslant \rho_2 \leqslant \cdots\,, \end{aligned} \tag{1.3.148}$$

there exist $\gamma, c > 0$ such that for $R > R_0$ and $\lambda_k < \gamma$,

$$|\lambda_k^2 - \rho_k| \leqslant e^{-cR}\,. \tag{1.3.149}$$

Proof. If $A, B \subseteq \mathbb{R}$ are symmetric (i.e., $\lambda \in A$ implies $-\lambda \in A$), then $\mathscr{E}_{A,R}(Z_{12,\infty}, F)$ and $\mathscr{E}_B(D^F_{Z_R})$ are homogeneous. Let $\mathscr{E}^p_{A,R}(Z_{12,\infty}, F)$ and $\mathscr{E}^p_B(D^F_{Z_R})$ be their degree p components. Noticing that $\mathscr{J}_{A,B,Z_R}$ preserves the degree, we denote by $\mathscr{J}^{(p)}_{A,B,Z_R}$ be the restriction of $\mathscr{J}_{A,B,Z_R}$ to $\mathscr{E}^p_{A,R}(Z_{12,\infty}, F)$. Then Proposition 1.3.13 and Proposition 1.3.17 hold for $\mathscr{J}^{(p)}_{A,B,Z_R}$. Noticing the fact that

$$\Big\{\lambda > 0 \ : \ \mathscr{E}^p_{\{\lambda,-\lambda\},R}(Z_{12,\infty}, F) \neq 0\Big\} = \Lambda^p_R \tag{1.3.150}$$

the rest of the proof follows the same argument as the proof of Theorem 1.3.18. □

1.4. **Asymptotic properties of the spectrum : boundary case.**

We use the notations in §1.3.1. We recall that the Riemannian manifolds $Z_{j,R} = Z_j \cup_Y [0, R] \times Y$ ($j = 1, 2$, $0 \leqslant R < \infty$) were defined in §1.0.2, and F is a flat vector bundle on $Z_{j,R}$. As stated in §1.0.2, we use the relative boundary condition on $\partial Z_{1,R}$ and the absolute boundary condition on $\partial Z_{2,R}$, which are defined by (1.1.5). We recall that $D^F_{Z_{j,R}}$ ($j = 1, 2$) are Hodge-de Rham operators acting $\Omega^\bullet_{\mathbf{bd}}(Z_{j,R}, F)$. Let $\mathrm{Sp}\left(D^F_{Z_{j,R}}\right)$ be the spectrum of $D^F_{Z_{j,R}}$. In this section, we give parallel results as in §1.3 for $\mathrm{Sp}\left(D^F_{Z_{j,R}}\right)$.

In §1.4.1, we establish results parallel to §1.3.3 and §1.3.4.

1.4.1. *Approximating the kernel and small eigenvalues.*

We recall that $\mathscr{H}^\bullet(Z_{j,\infty},F)$ and $\mathscr{H}^\bullet_{\mathrm{abs/rel}}(Z_{j,\infty},F) \subseteq \mathscr{H}^\bullet(Z_{j,\infty},F)$ ($j=1,2$) are defined by (1.2.35) and (1.2.48). We use the convention $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{1,\infty},F) = \mathscr{H}^\bullet_{\mathrm{rel}}(Z_{1,\infty},F)$ and $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{2,\infty},F) = \mathscr{H}^\bullet_{\mathrm{abs}}(Z_{2,\infty},F)$.

We recall that $I_{j,R}Y \subseteq Z_{j,R}$ ($j=1,2$) are the cylindrical parts of $Z_{j,R}$, defined in §1.3.1. We recall that the following maps are defined in Definition 1.2.9,

$$\mathscr{R}_{d^F} : \mathscr{H}^\bullet(Z_{j,\infty},F) \to \Omega^\bullet(I_{j,\infty}Y,F)\ , \quad \text{for } j=1,2\ . \tag{1.4.1}$$

The inclusion $I_{j,R}Y \subseteq I_{j,\infty}Y$ induces

$$\Omega^\bullet(I_{j,\infty}Y,F) \to \Omega^\bullet(I_{j,R}Y,F)\ . \tag{1.4.2}$$

Composing (1.4.1) and (1.4.2), we get

$$\mathscr{R}_{d^F,j} :\ \mathscr{H}^\bullet(Z_{j,\infty},F) \to \Omega^\bullet(I_{j,R}Y,F)\ , \quad \text{for } j=1,2\ . \tag{1.4.3}$$

We recall that $\chi_{j,R}$ ($j=1,2$) are defined by (1.3.24), which are smooth functions on I_RY. By restricting to $I_{j,R}Y \subseteq I_RY$ ($j=1,2$), we may view $\chi_{j,R}$ as smooth functions on $I_{j,R}Y$.

Parallel to Definition 1.3.1, we have the following definition.

Definition 1.4.1. We define

$$F_{Z_{j,R}} : \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F) \to \Omega^\bullet_{\mathbf{bd}}(Z_{j,R},F) \tag{1.4.4}$$

as follows: for $(\omega,\hat{\omega}) \in \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$,

$$F_{Z_{j,R}}(\omega,\hat{\omega})\big|_{Z_{j,0}} = \omega\ , \quad F_{Z_{j,R}}(\omega,\hat{\omega})\big|_{I_{j,R}Y} = d^F\Big(\chi_{j,R}\,\mathscr{R}_{d^F,j}(\omega,\hat{\omega})\Big) + \pi_Y^*\hat{\omega}\ . \tag{1.4.5}$$

By (1.2.42), $F_{Z_{j,R}}$ is well-defined. Furthermore, we have

$$d^F F_{Z_{j,R}}(\omega,\hat{\omega}) = 0\ . \tag{1.4.6}$$

We recall that $\varphi_R : Z_R \to Z$ is defined by (1.3.32). Put

$$\varphi_{j,R} = \varphi_R\big|_{Z_{j,R}} : Z_{j,R} \to Z_j\ . \tag{1.4.7}$$

Then $\varphi_{j,R}$ ($j=1,2$) induce the canonical isomorphisms $H^\bullet_{\mathbf{bd}}(Z_{j,R},Y) \simeq H^\bullet_{\mathbf{bd}}(Z_j,F)$.

Proposition 1.4.2. *For $R > R' > 0$ and $\omega_1 \in \mathscr{H}^\bullet_{L^2}(Z_{1,\infty},F)$, we have*

$$\big[F_{Z_{1,R}}(\omega_1,0)\big] = \big[F_{Z_{1,R'}}(\omega_1,0)\big] \in H^\bullet_{\mathbf{bd}}(Z_1,F)\ . \tag{1.4.8}$$

For $R > R' > 0$ and $(\omega_2,\hat{\omega}) \in \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{2,\infty},F)$, we have

$$\big[F_{Z_{2,R}}(\omega_2,\hat{\omega})\big] = \big[F_{Z_{2,R'}}(\omega_2,\hat{\omega})\big] \in H^\bullet_{\mathbf{bd}}(Z_2,F)\ . \tag{1.4.9}$$

We will prove Proposition 1.4.2 as a consequence of Proposition 1.3.3. We need the following constructions.

Let $\overline{Z}_{j,R}$ ($j=1,2$) be another copy of $Z_{j,R}$. Set $Z^{\mathrm{db}}_{j,R} = Z_{j,R} \cup_Y \overline{Z}_{j,R}$, which is a closed manifold. Then $Z^{\mathrm{db}}_{j,R}$ is equipped with a natural $\mathbb{Z}_2$-action exchanging $Z_{j,R}$ and $\overline{Z}_{j,R}$. Gluing the flat vector bundle F on $Z_{j,R}$ and its copy on $\overline{Z}_{j,R}$, we get a flat vector bundle on $Z^{\mathrm{db}}_{j,R}$, which is still denoted by F. The $\mathbb{Z}_2$-action lifts to F in the natural way. Let ι be the generator of this $\mathbb{Z}_2$-action. Gluing h^F and ι_*h^F, we get a Hermitian metric on F over $Z^{\mathrm{db}}_{j,R}$, which is still denoted by h^F. Let $D^F_{Z^{\mathrm{db}}_{j,R}}$ be the Hodge-de Rham operator acting on $\Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)$. Then $D^F_{Z^{\mathrm{db}}_{j,R}}$ is $\mathbb{Z}_2$-equivariant.

Let ι^* be the action on $\Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)$ or $H^\bullet(Z^{\mathrm{db}}_{j,R},F)$ induced by ι. Let $\big(\Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)\big)^\pm$ and $\big(H^\bullet(Z^{\mathrm{db}}_{j,R},F)\big)^\pm$ be its eigenspaces associated with ± 1. The injection $Z_{j,R}\hookrightarrow Z^{\mathrm{db}}_{j,R}$ induces the following isomorphism

$$\big(\Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)\big)^{(-1)^j}\to\Omega^\bullet_{\mathbf{bd}}(Z_{j,R},F)\;. \tag{1.4.10}$$

Passing to cohomology, we get the isomorphism

$$\big(H^\bullet(Z^{\mathrm{db}}_{j,R},F)\big)^{(-1)^j}\to H^\bullet_{\mathbf{bd}}(Z_{j,R},F)\;. \tag{1.4.11}$$

Proof of Proposition 1.4.2. Let $\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)$ be the $\mathscr{H}^\bullet(Z_{12,\infty},F)$ defined in §1.3.2 with $Z_{1,\infty}$ and $Z_{2,\infty}$ replaced by $Z_{j,\infty}$ and $\overline{Z}_{j,\infty}$. More precisely,

$$\begin{aligned}\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)=\Big\{(\omega_1,\omega_2,\hat\omega)\ :\ &(\omega_1,\hat\omega)\in\mathscr{H}^\bullet(Z_{j,\infty},F)\;,\\ &(\omega_2,\hat\omega)\in\mathscr{H}^\bullet(\overline{Z}_{j,\infty},F)\Big\}\;.\end{aligned} \tag{1.4.12}$$

By Definition 1.3.1, we have

$$F_{Z^{\mathrm{db}}_{j,R}}:\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)\to\Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)\;. \tag{1.4.13}$$

Let N^{du} be the number operator on $\mathscr{H}^\bullet(Y,F[du])$ associated to the variable du, i.e., its restriction to $\mathscr{H}^\bullet(Y,F)$ is zero, its restriction to $\mathscr{H}^\bullet(Y,F)du$ is the identity map. We define an involution

$$\begin{aligned}\iota^{\mathscr{H}}:\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)&\to\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)\\ (\omega_1,\omega_2,\hat\omega)&\mapsto(\omega_2,\omega_1,(-1)^{N^{du}}\hat\omega)\;.\end{aligned} \tag{1.4.14}$$

The following diagram commutes

$$\begin{tikzcd}\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F) \arrow[r, "\iota^{\mathscr{H}}"] \arrow[d, "F_{Z^{\mathrm{db}}_{j,R}}"] & \mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F) \arrow[d, "F_{Z^{\mathrm{db}}_{j,R}}"] \\ \Omega^\bullet(Z^{\mathrm{db}}_{j,R},F) \arrow[r, "\iota^*"] & \Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)\end{tikzcd}\;. \tag{1.4.15}$$

Let $\big(\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty}\big)^\pm$ be the eigenspace of $\iota^{\mathscr{H}}$ associated with ±1. We have

$$F_{Z^{\mathrm{db}}_{j,R}}:\big(\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)\big)^\pm\to\big(\Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)\big)^\pm\;. \tag{1.4.16}$$

We also have the following isomorphisms

$$\begin{aligned}\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)&\to\big(\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)\big)^{(-1)^j}\\ (\omega,\hat\omega)&\mapsto(\omega,(-1)^j\omega,\hat\omega)\;.\end{aligned} \tag{1.4.17}$$

The following diagram commutes

$$\begin{tikzcd}\big(\mathscr{H}^\bullet(Z^{\mathrm{db}}_{j,\infty},F)\big)^{(-1)^j} \arrow[r, "F_{Z^{\mathrm{db}}_{j,R}}"] \arrow[d] & \big(\Omega^\bullet(Z^{\mathrm{db}}_{j,R},F)\big)^{(-1)^j} \arrow[d] \\ \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F) \arrow[r, "F_{Z_{j,R}}"] & \Omega^\bullet_{\mathbf{bd}}(Z_{j,R},F)\end{tikzcd}\;, \tag{1.4.18}$$

where the vertical map on the left is defined by (1.4.17), the vertical map on the right is induced by the injection $Z_{j,R}\hookrightarrow Z^{\mathrm{db}}_{j,R}$.

By (1.4.11) and (1.4.18), the present proposition follows from Proposition 1.3.3 with Z_R replaced by $Z^{\mathrm{db}}_{j,R}$. □

In the rest of this section, we will state several results parallel to those in §1.3.3 and §1.3.4. Their proofs follow the same strategy as in the proof of Proposition 1.4.2 : on $Z^{\mathrm{db}}_{j,R}$, the constructions in §1.3 commute with the action of ι, and the objects concerned associated with $Z_{j,R}$ (eigenspace of Hodge-de Rham operator, cohomology, etc.) are canonically isomorphic to the eigenspaces of ι associated with $(-1)^j$ in the corresponding objects associated with $Z^{\mathrm{db}}_{j,R}$.

Recall that the L^2-norm $\|\cdot\|_.$ is defined in §1.0.4. For $(\omega,\hat{\omega})\in\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$, put

$$\|(\omega,\hat{\omega})\|^2_{\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F),R}=\|\omega\|^2_{Z_{j,R}}\ . \tag{1.4.19}$$

By passing to $Z^{\mathrm{db}}_{j,R}$ and applying Proposition 1.3.4, we get the following proposition.

Proposition 1.4.3. *There exist $c>0$, $R_0>0$ such that for any $R>R_0$, $(\omega,\hat{\omega})\in\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ $(j=1,2)$, we have*

$$1-e^{-cR}\leqslant\frac{\left\|F_{Z_{j,R}}(\omega,\hat{\omega})\right\|_{Z_{j,R}}}{\|(\omega,\hat{\omega})\|_{\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F),R}}\leqslant 1+e^{-cR}\ . \tag{1.4.20}$$

Let

$$P^{\ker(D^{F,2}_{Z_{j,R}})}:\Omega^\bullet_{\mathbf{bd}}(Z_{j,R},F)\to\ker\left(D^{F,2}_{Z_{j,R}}\right) \tag{1.4.21}$$

be the orthogonal projections.

Definition 1.4.4. For $j=1,2$, set

$$\mathscr{F}_{Z_{j,R}}=P^{\ker(D^{F,2}_{Z_{j,R}})}\circ F_{Z_{j,R}}:\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)\to\ker\left(D^{F,2}_{Z_{j,R}}\right)\ . \tag{1.4.22}$$

By passing to $Z^{\mathrm{db}}_{j,R}$ and applying Proposition 1.3.7, we get the following proposition.

Proposition 1.4.5. *There exist $c>0$, $R_0>0$ such that for any $R>R_0$, $(\omega,\hat{\omega})\in\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ $(j=1,2)$, we have*

$$\left\|(F_{Z_{j,R}}-\mathscr{F}_{Z_{j,R}})(\omega,\hat{\omega})\right\|_{\mathscr{C}^0,Z_{j,R}}\leqslant e^{-cR}\|(\omega,\hat{\omega})\|_{\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)}\ . \tag{1.4.23}$$

By passing to $Z^{\mathrm{db}}_{j,R}$ and applying Theorem 1.3.10, we get the following theorem.

Theorem 1.4.6. *There exists $R_0>0$ such that for $R>R_0$, the maps $\mathscr{F}_{Z_{j,R}}$ $(j=1,2)$ is bijective. Moreover,*

$$\mathrm{Sp}\left(D^F_{Z_{j,R}}\right)\subseteq\,]-\infty,-R^{-1-\varepsilon}[\,\cup\{0\}\cup\,]R^{-1-\varepsilon},+\infty[\ . \tag{1.4.24}$$

Set

$$C_{j,\mathbf{bd}}(\lambda)=(-1)^j\left(C_j(\lambda)\big|_{\mathscr{H}^\bullet(Y,F)}-C_j(\lambda)\big|_{\mathscr{H}^\bullet(Y,F)du}\right)\ . \tag{1.4.25}$$

For $R\geqslant 0$, set

$$\Lambda_{j,R}=\left\{\lambda\in\mathbb{R}\ ,\ \det\left(e^{2i\lambda R}C_{j,\mathbf{bd}}(\lambda)\big|_{\mathscr{H}^\bullet(Y,F)}-1\right)=0\right\}\ ,\quad\text{for } j=1,2\ . \tag{1.4.26}$$

By passing to $Z^{\mathrm{db}}_{j,R}$ and applying Theorem 1.3.18, we get the following theorem.

Theorem 1.4.7. *Theorem 1.3.18 holds for $\left(\mathrm{Sp}\left(D^F_{Z_{j,R}}\right),\Lambda_{j,R}\right)$, where $j=1,2$.*

For $0 \leqslant p \leqslant \dim Z$, set

$$C^p_{j,\mathbf{bd}}(\lambda) = C_{j,\mathbf{bd}}(\lambda)\big|_{\mathscr{H}^p(Y,F)\oplus\mathscr{H}^{p-1}(Y,F)du}\,, \quad \text{for } j=1,2\,,$$
$$\Lambda^p_{j,R} = \Big\{\lambda\in\mathbb{R}\,,\ \det\Big(e^{2i\lambda R}C^p_{j,\mathbf{bd}}(\lambda)\big|_{\mathscr{H}^p(Y,F)}-1\Big)=0\Big\}\,. \tag{1.4.27}$$

By passing to $Z^{\mathrm{db}}_{j,R}$ and applying Theorem 1.3.19, we get the following theorem.

Theorem 1.4.8. *Theorem 1.3.19 holds for* $\Big(\mathrm{Sp}\Big(D^{F,2,(p)}_{Z_{j,R}}\Big)\,,\Lambda^p_{j,R}\Big)$*, where* $j=1,2$.

1.5. **Asymptotics of the (weighted) zeta determinants.**

The purpose of this section is to prove Theorem 1.0.1.

In this section, we use notations in §1.3.1. For convenience, we use the following convention : $Z_{0,R}=Z_R$, $\zeta_{0,R}=\zeta_R$, and so forth, i.e., we add a sub-index 0 to the objects associated with Z_R. We use the following definition of ζ-functions $\zeta_{j,R}(s)$ $(j=0,1,2)$, which is equivalent to (1.0.6).

$$\zeta_{j,R}(s) = -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}\Big[(-1)^N N\exp\Big(-tD^{F,2}_{Z_{j,R}}\Big)\Big(1-P^{\ker(D^{F,2}_{Z_{j,R}})}\Big)\Big]\,dt\,. \tag{1.5.1}$$

Let $\varepsilon\in\,]0,1[$. Let $\zeta^S_{j,R}(s)$ (resp. $\zeta^L_{j,R}(s)$) be the contribution of $\int_0^{R^{2-\varepsilon}}$ (resp. $\int_{R^{2-\varepsilon}}^\infty$) to $\zeta_{j,R}(s)$ in (1.5.1). Then

$$\zeta_{j,R} = \zeta^S_{j,R}+\zeta^L_{j,R}\,. \tag{1.5.2}$$

In §1.5.1, we define model operators which will serve as the limit (as $R\to\infty$) of the Hodge-de Rham operators concerned. In §1.5.2, we treat the contributions of $\zeta^S_{j,R}$. In §1.5.3, we treat the contributions of $\zeta^L_{j,R}$.

1.5.1. *Model operators.*

Set $I_{1,R}=[-R,0]$, $I_{2,R}=[0,R]$ and $I_R=[-R,R]$. Let u be the coordinate. We sometimes add a sub-index 0 to the objects associated with $I_{0,R}:=I_R$.

We recall that $\mathscr{H}^\bullet(Y,F)$ and $\mathscr{H}^\bullet(Y,F[du])$ are defined by (1.2.1) and (1.2.6). Let $\Omega^\bullet\big(I_R,\mathscr{H}^\bullet(Y,F)\big)$ be the vector space of differential forms on I_R with values in $\mathscr{H}^\bullet(Y,F)$. We define the total degree of $\omega\in\Omega^p(I_R,\mathscr{H}^q(Y,F))$ to be $p+q$. We have the canonical identification

$$\Omega^\bullet\big(I_R,\mathscr{H}^\bullet(Y,F)\big)\simeq\mathscr{C}^\infty\big(I_R,\mathscr{H}^\bullet(Y,F[du])\big)\,. \tag{1.5.3}$$

For $\omega\in\Omega^\bullet\big(I_R,\mathscr{H}^\bullet(Y,F)\big)$, let $u\mapsto\omega_u\in\mathscr{H}^\bullet(Y,F[du])$ be the corresponding function.

We recall that the operator $c(\frac{\partial}{\partial u})$ acting on $\mathscr{H}^\bullet(Y,F[du])$ is defined by (1.2.4) and that $\mathscr{L}^\bullet_j\subseteq\mathscr{H}^\bullet(Y,F[du])$ $(j=1,2)$ are defined at the begining of §1.3.2. We define the model operator D_{I_R} by

$$D_{I_R}=c(\tfrac{\partial}{\partial u})\frac{\partial}{\partial u}\,, \tag{1.5.4}$$

with

$$\mathrm{Dom}\,(D_{I_R})=\Big\{\omega\in\Omega^\bullet(I_R,\mathscr{H}^\bullet(Y,F))\ :\ \omega_{-R}\in\mathscr{L}^\bullet_1\,,\ \omega_R\in\mathscr{L}^\bullet_2\Big\}\,. \tag{1.5.5}$$

We define equally $D_{I_{1,R}}$ and $D_{I_{2,R}}$ with

$$\begin{aligned}\mathrm{Dom}\left(D_{I_{1,R}}\right) &= \Big\{\omega\in\Omega^\bullet(I_{1,R},\mathscr{H}^\bullet(Y,F))\ :\ \omega_{-R}\in\mathscr{L}_1^\bullet\ ,\\ &\qquad\qquad \omega_0\in\mathscr{H}^\bullet(Y,F)du\Big\}\ ,\\ \mathrm{Dom}\left(D_{I_{2,R}}\right) &= \Big\{\omega\in\Omega^\bullet(I_{2,R},\mathscr{H}^\bullet(Y,F))\ :\ \omega_{R}\in\mathscr{L}_2^\bullet\ ,\\ &\qquad\qquad \omega_0\in\mathscr{H}^\bullet(Y,F)\Big\}\ .\end{aligned}\tag{1.5.6}$$

We remark that $D^2_{I_{j,R}}$ $(j=0,1,2)$ preserve the total degree. Let $D^{2,(p)}_{I_{j,R}}$ be its restriction to total degree p.

Let $\mathscr{L}^\bullet_{j,\mathrm{abs/rel}}$ be absolute/relative part of $\mathscr{L}^\bullet_j$, which is defined by (1.2.46). We use the convention $\mathscr{L}^\bullet_{1,\mathbf{bd}}=\mathscr{L}^\bullet_{1,\mathrm{rel}}$ and $\mathscr{L}^\bullet_{2,\mathbf{bd}}=\mathscr{L}^\bullet_{2,\mathrm{abs}}$. By (1.5.4), (1.5.5) and (1.5.6), we have

$$\ker\left(D^{2,(p)}_{I_R}\right)=\mathscr{L}_1^p\cap\mathscr{L}_2^p\ ,\quad \ker\left(D^{2,(p)}_{I_{j,R}}\right)=\mathscr{L}^p_{j,\mathbf{bd}}\ ,\quad \text{for } j=1,2\ ,\tag{1.5.7}$$

where the vectors in $\mathscr{L}_1^p\cap\mathscr{L}_2^p$ (resp. $\mathscr{L}^p_{j,\mathbf{bd}}$) are viewed as constant functions on I_R (resp. $I_{j,R}$).

We define the composition map

$$\alpha_{p,\mathscr{L}}:\mathscr{L}^p_{1,\mathrm{rel}}\to\mathscr{L}^p_{1,\mathrm{rel}}\cap\mathscr{L}^p_{2,\mathrm{rel}}\to\mathscr{L}_1^p\cap\mathscr{L}_2^p\ ,\tag{1.5.8}$$

where the first map is the orthogonal projection, and the second one is the natural injection. We also define

$$\beta_{p,\mathscr{L}}:\mathscr{L}_1^p\cap\mathscr{L}_2^p\to\mathscr{L}^p_{1,\mathrm{abs}}\cap\mathscr{L}^p_{2,\mathrm{abs}}\to\mathscr{L}^p_{2,\mathrm{abs}}\ ,\tag{1.5.9}$$

which is still the composition of an orthogonal projection and an injection. And

$$\delta_{p,\mathscr{L}}:\mathscr{L}^p_{2,\mathrm{abs}}\to\mathscr{L}^{p+1,\perp}_{2,\mathrm{rel}}\to\mathscr{L}^{p+1}_{1,\mathrm{rel}}\cap\mathscr{L}^{p+1,\perp}_{2,\mathrm{rel}}\to\mathscr{L}^{p+1}_{1,\mathrm{rel}}\ ,\tag{1.5.10}$$

where the first map is the $du\wedge$ operation (cf. (1.2.4)), the second one is the orthogonal projection and the last one is the natural injection. We get the following exact sequence

$$\cdots\longrightarrow\mathscr{L}^p_{1,\mathbf{bd}}\xrightarrow{\alpha_{p,\mathscr{L}}}\mathscr{L}_1^p\cap\mathscr{L}_2^p\xrightarrow{\beta_{p,\mathscr{L}}}\mathscr{L}^p_{2,\mathbf{bd}}\xrightarrow{\delta_{p,\mathscr{L}}}\cdots\ .\tag{1.5.11}$$

The exactness of (1.5.11) is justified by the following identities

$$\begin{aligned}\mathrm{im}(\alpha_{p,\mathscr{L}})&=\ker(\beta_{p,\mathscr{L}})=\mathscr{L}^p_{1,\mathrm{rel}}\cap\mathscr{L}^p_{2,\mathrm{rel}}\ ,\\ \mathrm{im}(\beta_{p,\mathscr{L}})&=\ker(\delta_{p,\mathscr{L}})=\mathscr{L}^p_{1,\mathrm{abs}}\cap\mathscr{L}^p_{2,\mathrm{abs}}\ ,\\ \mathrm{im}(\delta_{p,\mathscr{L}})&=\ker(\alpha_{p+1,\mathscr{L}})=\mathscr{L}^{p+1}_{1,\mathrm{rel}}\cap\mathscr{L}^{p+1,\perp}_{2,\mathrm{rel}}\ .\end{aligned}\tag{1.5.12}$$

We may view (1.5.11) as the Mayer-Vietoris exact sequence associated with our model.

Recall that $C_j(\lambda)\in\mathrm{End}\big(\mathscr{H}^\bullet(Y,F[du])\big)$ $(j=1,2)$ are the scattering matrices associated with $D^F_{Z_{j,\infty}}$ (cf. §1.3.2) and that the operators $C_{12}(\lambda)$ and $C_{j,\mathbf{bd}}(\lambda)$ are introduced in (1.3.18) and (1.4.25). We denote $C_{12}=C_{12}(0)$ (resp. $C_{j,\mathbf{bd}}=C_{j,\mathbf{bd}}(0)$). Let C^p_{12} (resp.$C^p_{j,\mathbf{bd}}$) be its restriction to $\mathscr{H}^p(Y,F)\oplus\mathscr{H}^{p-1}(Y,F)du$.

By (1.2.45) and (1.2.46), we have

$$
\begin{aligned}
\ker\left(C^p_{1,\mathbf{bd}}-1\right) &= \mathscr{L}^p_{1,\mathrm{rel}} \oplus i_{\frac{\partial}{\partial u}} \mathscr{L}^{p+1}_{1,\mathrm{rel}} \,, \\
\ker\left(C^p_{2,\mathbf{bd}}-1\right) &= \mathscr{L}^p_{2,\mathrm{abs}} \oplus du \wedge \mathscr{L}^{p-1}_{2,\mathrm{abs}} \,, \\
\ker\left(C^p_{12}-1\right) &= \left(\mathscr{L}^p_1 \cap \mathscr{L}^p_2\right) \oplus i_{\frac{\partial}{\partial u}} \left(\mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap \mathscr{L}^{p+1}_{2,\mathrm{rel}}\right) \\
&\qquad \oplus du \wedge \left(\mathscr{L}^{p-1}_{1,\mathrm{abs}} \cap \mathscr{L}^{p-1}_{2,\mathrm{abs}}\right) .
\end{aligned} \tag{1.5.13}
$$

For $C = C_{12}$ or $C_{j,\mathbf{bd}}$ $(j=1,2)$, set

$$
\chi'(C) = \sum_p (-1)^p p \dim\ker\left(C^p - 1\right) . \tag{1.5.14}
$$

We recall that χ' is defined in (1.0.9).

Lemma 1.5.1. *We have*

$$
\chi'(C_{12}) - \chi'(C_{1,\mathbf{bd}}) - \chi'(C_{2,\mathbf{bd}}) = 2\chi'. \tag{1.5.15}
$$

Proof. We denote

$$
\begin{aligned}
&\dim \mathscr{L}^p_{1,\mathrm{abs}} = x_p \,, \quad \dim \mathscr{L}^p_{2,\mathrm{abs}} = y_p \,, \\
&\dim(\mathscr{L}^p_{1,\mathrm{abs}} \cap \mathscr{L}^p_{2,\mathrm{abs}}) = u_p \,, \quad \dim(\mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap \mathscr{L}^{p,\perp}_{2,\mathrm{abs}}) = v_p \,, \\
&\dim \mathscr{H}^p(Y,F) = h_p \,.
\end{aligned} \tag{1.5.16}
$$

Then, by (1.2.46), we have

$$
\begin{aligned}
&\dim \mathscr{L}^{p+1}_{1,\mathrm{rel}} = h_p - x_p \,, \quad \dim \mathscr{L}^{p+1}_{2,\mathrm{rel}} = h_p - y_p \,, \\
&\dim(\mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap \mathscr{L}^{p+1}_{2,\mathrm{rel}}) = v_p \,.
\end{aligned} \tag{1.5.17}
$$

Since $\mathscr{H}^p(Y,F) = (\mathscr{L}^p_{1,\mathrm{abs}} + \mathscr{L}^p_{2,\mathrm{abs}}) \oplus (\mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap \mathscr{L}^{p,\perp}_{2,\mathrm{abs}})$, we get

$$
h_p = x_p + y_p - u_p + v_p \,. \tag{1.5.18}
$$

By (1.5.13), (1.5.14), (1.5.17) and (1.5.18), we have

$$
\begin{aligned}
\chi'(C_{12}) - \chi'(C_{1,\mathbf{bd}}) - \chi'(C_{2,\mathbf{bd}}) &= \sum_p 2(-1)^p (y_p - u_p) \,, \\
\dim \mathscr{L}^p_1 \cap \mathscr{L}^p_2 - \dim \mathscr{L}^p_{1,\mathbf{bd}} - \dim \mathscr{L}^p_{2,\mathbf{bd}} &= \sum_p (-1)^p (y_p - u_p) \,.
\end{aligned} \tag{1.5.19}
$$

By (1.0.9) and (1.5.19), it rests to show that

$$
\begin{aligned}
&\dim \mathscr{L}^p_1 \cap \mathscr{L}^p_2 - \dim \mathscr{L}^p_{1,\mathbf{bd}} - \dim \mathscr{L}^p_{2,\mathbf{bd}} \\
&= \dim H^p(Z,F) - \dim H^p_{\mathbf{bd}}(Z_1,F) - \dim H^p_{\mathbf{bd}}(Z_2,F) \,.
\end{aligned} \tag{1.5.20}
$$

By Theorem 1.1.1, Theorem 1.3.10 and Theorem 1.4.6, (1.5.20) is equivalent to

$$
\begin{aligned}
&\dim \mathscr{L}^p_1 \cap \mathscr{L}^p_2 - \dim \mathscr{L}^p_{1,\mathbf{bd}} - \dim \mathscr{L}^p_{2,\mathbf{bd}} \\
&= \dim \mathscr{H}^p(Z_{12,\infty},F) - \dim \mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty},F) - \dim \mathscr{H}^p_{\mathbf{bd}}(Z_{2,\infty},F) \,.
\end{aligned} \tag{1.5.21}
$$

This follows from (1.2.49) and (1.3.13). □

We denote

$$
a_p = \dim \mathrm{im}(\alpha_{p,\mathscr{L}}) \,, \quad b_p = \dim \mathrm{im}(\beta_{p,\mathscr{L}}) \,, \quad d_p = \dim \mathrm{im}(\delta_{p,\mathscr{L}}) \,. \tag{1.5.22}
$$

Lemma 1.5.2. *We have*

$$\chi' = \sum_p (-1)^p d_p \ , \qquad \chi'(C_{12}) = \sum_p (-1)^p (a_p - b_p) \ . \tag{1.5.23}$$

Proof. Proceeding in the same way as in the proof of Lemma 1.5.1, all the terms involved can be expressed by x_p, y_p, u_p, v_p. Then (1.5.23) follows from a direct calculation. □

We turn to study the spectra and ζ-functions associated with our model.

For $R \geqslant 0$, set

$$\begin{aligned} \Lambda_R^{*,p} &= \left\{ \lambda > 0 \ : \ \det\left(e^{4i\lambda R} C_{12}^p - 1\right) = 0 \right\} , \\ \Lambda_{j,R}^{*,p} &= \left\{ \lambda > 0 \ : \ \det\left(e^{2i\lambda R} C_{j,\mathbf{bd}}^p - 1\right) = 0 \right\} , \quad \text{for } j = 1, 2 \ . \end{aligned} \tag{1.5.24}$$

Proposition 1.5.3. *We have*

$$\mathrm{Sp}\left(D_{I_{j,R}}^{2,(p)}\right) \backslash \{0\} = \left\{ \lambda^2 \ : \ \lambda \in \Lambda_{j,R}^{*,p} \right\} , \quad \textit{for } j = 0, 1, 2 \ . \tag{1.5.25}$$

Proof. First we consider the case $j = 0$.

By shifting the coordinate, we identify $I_{1,R}$ to $[0, R]$. We define $I_{1,\infty} = [0, \infty[$. Let $D_{I_{1,\infty}}$ be the operator defined by (1.5.4) with the same boundary condition (only at $u = 0$) as $D_{I_{1,R}}$ for $R < \infty$. Here, $D_{I_{1,\infty}}$ is exactly the $D_{Z_\infty}^F$ constructed in §1.2.3 with Z_∞ replaced by $I_{1,\infty}$ and F replaced by $\mathscr{H}^\bullet(Y, F)$. Using (1.2.45) and (1.2.46), a direct calculation shows that a generalized eigensection of $D_{I_{1,\infty}}$ with eigenvalue $\lambda \neq 0$ takes the following form

$$e^{-i\lambda u}(1 - ic(\tfrac{\partial}{\partial u}))\phi + e^{i\lambda u} C_1 (1 - ic(\tfrac{\partial}{\partial u}))\phi \ , \quad \phi \in \mathscr{H}^\bullet(Y, F) \ . \tag{1.5.26}$$

Comparing to (1.2.31), we see that there are only zeromodes (cf. (1.2.14), (1.2.15)). Furthermore, the scattering matrix of $D_{I_{1,\infty}}$ is C_1, which does not depend on λ.

We construct equally $D_{I_{2,\infty}}$. Its scattering matrix is C_2.

With the above constructions, we are in a special case of the problem addressed in §1.3. The only difference is that I_R is not a closed manifold. Checking all the arguments in §1.3, we see that they still work for D_{I_R}. Now, applying Theorem 1.3.19, we see that $\mathrm{Sp}\left(D_{I_R}^{2,(p)}\right) \backslash \{0\}$ is approximated by $\Lambda_R^{*,p}$ in the sense of (1.3.149). Notice that the error terms in the whole argument leading to Theorem 1.3.19 come from non zeromodes. Here, since there are only zeromodes, the approximation is replaced by equality. This proves (1.5.25).

For $j = 1, 2$, replacing Theorem 1.3.19 by Theorem 1.4.7, the same argument works. □

Let $\zeta_{*,j,R}(s)$ be the ζ-functions of $D_{I_{j,R}}^2$ defined in the same way as (1.5.1) .

Proposition 1.5.4. *We have*

$$\begin{aligned} {\zeta_{*,R}}'(0) &= \chi'(C_{12}) \log(2R) - \chi(Y, F) \log 2 \\ &\quad + \sum_{p=0}^{\dim Y} \frac{p}{2} (-1)^p \log\det{}^* \left(\frac{2 - C_{12}^p - (C_{12}^p)^{-1}}{4} \right) , \\ {\zeta_{*,j,R}}'(0) &= \chi'(C_{j,\mathbf{bd}}) \log R - \chi(Y, F) \log 2 \ , \quad \textit{for } j = 1, 2 \ . \end{aligned} \tag{1.5.27}$$

Proof. Applying (1.0.6) and (1.5.25), both identities are consequences of Appendix (1.8.16). The first identity is the weighted sum of (1.8.16) with V replaced by $\mathscr{H}^p(Y,F) \oplus \mathscr{H}^{p-1}(Y,F)du$ and C replaced by C^p_{12}. For the second identity, we replace C by $C^p_{j,\mathbf{bd}}$ and replace R by $R/2$. Since $\mathrm{Sp}\big(C^p_{j,\mathbf{bd}}\big) \subseteq \big\{-1,1\big\}$, the $\log\det^*$ term vanishes. □

1.5.2. *Small time contribution.*

We denote

$$\Theta_R(t) = \sum_{j=0}^{2} (-1)^{(j-1)(j-2)/2}\,\mathrm{Tr}\left[(-1)^N N \exp\left(-tD^{F,2}_{Z_{j,R}}\right)\right] ,$$
$$\Theta^*_R(t) = \sum_{j=0}^{2} (-1)^{(j-1)(j-2)/2}\,\mathrm{Tr}\left[(-1)^N N \exp\left(-tD^{2}_{I_{j,R}}\right)\right] . \tag{1.5.28}$$

We define $\zeta^{S/L}_{*,j,R}$ $(j=0,1,2)$ in the same way as $\zeta^{S/L}_{R}$.

By (1.5.1) and (1.5.15), we have

$$\begin{aligned}&\sum_{j=0}^{2} (-1)^{(j-1)(j-2)/2}\Big(\zeta^S_{j,R}(s) - \zeta^S_{*,j,R}(s)\Big)\\ &= -\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\big(\Theta_R(t) - \Theta^*_R(t)\big)dt\ .\end{aligned} \tag{1.5.29}$$

Theorem 1.5.5. *There exist $c>0$ such that as $R\to\infty$,*

$$\sum_{j=0}^{2} (-1)^{(j-1)(j-2)/2}\Big({\zeta^S_{j,R}}'(0) - {\zeta^S_{*,j,R}}'(0)\Big) = \mathscr{O}\big(e^{-cR^{\varepsilon/2}}\big)\ . \tag{1.5.30}$$

Proof. Let $f\in\mathscr{C}^\infty(\mathbb{R})$ be an even function such that $f(u)=1$ for $|u|\leqslant 1/2$ and $f(u)=0$ for $|u|\geqslant 1$. We proceed in the same way as in [BL91, §13(b)]. For $t,\varsigma>0$ and $z\in\mathbb{C}$, set

$$F_{t,\varsigma}(z) = \int_{-\infty}^{\infty} e^{i\sqrt{2}vz}e^{-\frac{1}{2}v^2} f\big(\sqrt{\varsigma t}v\big)\frac{dv}{\sqrt{2\pi}}\ ,$$
$$G_{t,\varsigma}(z) = \int_{-\infty}^{\infty} e^{i\sqrt{2}vz}e^{-\frac{1}{2}v^2/t}\left(1 - f\big(\sqrt{\varsigma}v\big)\right)\frac{dv}{\sqrt{2\pi t}}\ . \tag{1.5.31}$$

Then

$$F_{t,\varsigma}\big(\sqrt{t}z\big) + G_{t,\varsigma}\big(z\big) = \exp\big(-tz^2\big)\ . \tag{1.5.32}$$

Let

$$\begin{aligned}&F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_{j,R}}\right)(x,y)\ ,\ G_{t,\varsigma}\left(D^F_{Z_{j,R}}\right)(x,y)\\ &\qquad\in \big(\Lambda^\bullet\big(T^*Z_{j,R}\big)\otimes F\big)_x \otimes \big(\Lambda^\bullet\big(T^*Z_{j,R}\big)\otimes F\big)^*_y\end{aligned} \tag{1.5.33}$$

be the smooth kernel of operators $F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_{j,R}}\right)$ and $G_{t,\varsigma}\left(D^F_{Z_{j,R}}\right)$ with respect to the volume form induced by the Riemannian metric on $Z_{j,R}$.

By the construction of $G_{t,\varsigma}(z)$, for any $k\in\mathbb{N}$, there exists $c,C>0$ such that for any $t>0$ and $0<\varsigma<1$, we have (cf. [MaMar07, (1.6.16)])

$$\sup_{z\in\mathbb{C}}\big|z^kG_{t,\varsigma}(z)\big| \leqslant Ce^{-c/\varsigma t}\ . \tag{1.5.34}$$

As a consequence, for any $k,k'\in\mathbb{N}$, there exists $c,C>0$ such that for $0<t<R^{2-\varepsilon}$, $0<\varsigma<R^{-2+\varepsilon/2}$ and $j=0,1,2$, we have

$$\left\| D^{F,k}_{Z_{j,R}} G_{t,\varsigma}\left(D^F_{Z_{j,R}}\right) D^{F,k'}_{Z_{j,R}} \right\|_{0,0} \leqslant Cte^{-cR^{\varepsilon/2}} \ , \tag{1.5.35}$$

where $\|\cdot\|_{0,0}$ is the operator norm induced by the L^2-norm. By Proposition 1.3.5 and (1.5.35), there exists $c,C>0$ such that for $0<t<R^{2-\varepsilon}$, $0<\varsigma<R^{-2+\varepsilon/2}$, $j=0,1,2$ and $x,y\in Z_{j,R}$, we have

$$\left| G_{t,\varsigma}\left(D^F_{Z_{j,R}}\right)(x,y)\right| \leqslant Cte^{-cR^{\varepsilon/2}} \ . \tag{1.5.36}$$

By the finite propagation speed principal (cf. [T11, §2.6, Theorem 6.1], [MaMar07, Appendix D.2]), if the distance between x and y is greater than $\varsigma^{-1/2}$, $F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_{j,R}}\right)(x,y)=0$. In the rest of the proof, we take $\varsigma=R^{-2+\varepsilon/3}$ and suppose that R is large enough. For $x\in Z_{j,R/2}\subseteq Z_{j,R}\subseteq Z_R$ ($j=1,2$), we have

$$F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_{j,R}}\right)(x,x) = F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_R}\right)(x,x) \ . \tag{1.5.37}$$

We view the middle of the cylinder $]-R/2,R/2[\times Y$ as a subset of $\mathbb{R}\times Y$. Let $D^F_{\mathbb{R}Y}$ be the Hodge-de Rham operator acting on $\Omega^\bullet(\mathbb{R}\times Y,F)$. Let ι be the involution on $\mathbb{R}\times Y$ sending (u,y) to $(-u,y)$. For $x\in\big(]-R/2,R/2[\times Y\big)\cap Z_{j,R}$ ($j=1,2$), we have

$$\begin{aligned} & F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_{j,R}}\right)(x,x) \\ & = F_{t,\varsigma}\left(\sqrt{t}D^F_{\mathbb{R}Y}\right)(x,x) + (-1)^j F_{t,\varsigma}\left(\sqrt{t}D^F_{\mathbb{R}Y}\right)(x,\iota x) \ . \end{aligned} \tag{1.5.38}$$

As a consequence, for $x\in]-R/2,R/2[\times Y\cap Z_{1,R}=]-R/2,0]\times Y$, we have

$$\begin{aligned} & F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_{1,R}}\right)(x,x) + \iota^* F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_{2,R}}\right)(\iota x,\iota x) \\ & = F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_R}\right)(x,x) + \iota^* F_{t,\varsigma}\left(\sqrt{t}D^F_{Z_R}\right)(\iota x,\iota x) \\ & \in \mathrm{End}\left(\Lambda^\bullet\big(T^*Z_{j,R}\big)\otimes F\right)_x \ . \end{aligned} \tag{1.5.39}$$

By (1.5.32), $\Theta_R(t)$ can be decomposed to the contributions of $F_{t,\varsigma}$ and $G_{t,\varsigma}$. By (1.5.37) and (1.5.39), the contribution of $F_{t,\varsigma}$ to (1.5.29) vanishes identically. By (1.5.36), the contribution of $G_{t,\varsigma}$ to (1.5.29) together with its derivative at $s=0$ are $\mathscr{O}\big(e^{-R^{\varepsilon/2}}\big)$-small. For $\Theta^*_R(t)$, the same argument works. This terminates the proof of (1.5.30). □

1.5.3. *Large time contribution and proof of Theorem 1.0.1.*

By (1.5.1) and (1.5.15), we have

$$\begin{aligned} & \sum_{j=0}^{2}(-1)^{(j-1)(j-2)/2}\Big(\zeta^L_{j,R}(s)-\zeta^L_{*,j,R}(s)\Big) \\ & = -\frac{1}{\Gamma(s)}\int_{R^{2-\varepsilon}}^\infty t^{s-1}\big(\Theta_R(t)-\Theta^*_R(t)\big)dt \ . \end{aligned} \tag{1.5.40}$$

Let $\kappa\in]\varepsilon,1[$. Let $\Theta^{I}_R(t)$ (resp. $\Theta^{II}_R(t)$) be the contribution to $\Theta_R(t)$ by the eigenvalues of $D^{F,2}_{Z_{j,R}}$ ($j=0,1,2$) less than (resp. greater than or equal to) $R^{-2+\kappa}$. We define $\Theta^{*,I}_R(t)$ and $\Theta^{*,II}_R(t)$ in the same way.

Proposition 1.5.6. *As $R \to \infty$, we have*

$$\int_{R^{2-\varepsilon}}^{\infty} \Theta_R^{II}(t)\, \frac{dt}{t} = \mathscr{O}\left(e^{-\frac{1}{2}R^{\kappa-\varepsilon}}\right) \ , \quad \int_{R^{2-\varepsilon}}^{\infty} \Theta_R^{*,II}(t)\, \frac{dt}{t} = \mathscr{O}\left(e^{-\frac{1}{2}R^{\kappa-\varepsilon}}\right) \ . \tag{1.5.41}$$

Proof. Let $\{\lambda_k\}_k$ be the set of eigenvalues of $D_{Z_{j,R}}^{F,2}$ $(j = 0,1,2)$ such that $\lambda_k \geqslant R^{-2+\kappa}$. Let $n = \dim Z$. Then for R large and $t \geqslant R^{2-\varepsilon}$, we have

$$\begin{aligned} \left|\Theta_R^{II}(t)\right| & \leqslant n \sum_k e^{-t\lambda_k} \leqslant n e^{-(t-1)R^{-2+\kappa}} \sum_k e^{-\lambda_k} \\ & \leqslant n e^{-(t-1)R^{-2+\kappa}} \sum_{j=0}^{2} \operatorname{Tr}\left[\exp\left(-D_{Z_{j,R}}^{F,2}\right)\right] \ . \end{aligned} \tag{1.5.42}$$

Let $\exp\left(-D_{Z_{j,R}}^{F,2}\right)(x,y)$ $(x, y \in Z_{j,R})$ be the smooth kernel of the operator $\exp\left(-D_{Z_{j,R}}^{F,2}\right)$ with respect to the volume form induced by the Riemannian metric on $Z_{j,R}$. Proceeding in the same way as (1.5.36), there exists $C > 0$ such that for any $x, y \in Z_{j,R}$,

$$\left|\exp\left(-D_{Z_{j,R}}^{F,2}\right)(x,y)\right| \leqslant C \ . \tag{1.5.43}$$

As a consequence, there exist $a, b > 0$, such that

$$\operatorname{Tr}\left[\exp\left(-D_{Z_{j,R}}^{F,2}\right)\right] \leqslant a \operatorname{Vol}(Z_{j,R}) \leqslant bR \ , \quad \text{for } j = 0,1,2 \ . \tag{1.5.44}$$

By (1.5.42) and (1.5.44), we get the first estimate in (1.5.41). The second one can be established in the same way. □

Proposition 1.5.7. *As $R \to \infty$, we have*

$$\int_{R^{2-\varepsilon}}^{\infty} \left(\Theta_R^{I}(t) - \Theta_R^{*,I}(t)\right) \frac{dt}{t} = \mathscr{O}\left(R^{\kappa-1}\right) \tag{1.5.45}$$

Proof. For $\lambda > 0$, we denote

$$e_R(\lambda) = \int_{R^{2-\varepsilon}}^{\infty} e^{-t\lambda}\, \frac{dt}{t} = \int_{R^{2-\varepsilon}\lambda}^{\infty} e^{-t}\, \frac{dt}{t} \ . \tag{1.5.46}$$

By splitting the integral to $\int_1^\infty + \int_{R^{2-\varepsilon}\lambda}^1$ (if $R^{2-\varepsilon}\lambda \leqslant 1$), we have

$$\left|e_R(\lambda)\right| \leqslant 1 + \max\left\{ -\log\left(R^{2-\varepsilon}\lambda\right) \ , \ 0\right\} \ , \quad \left|e_R{}'(\lambda)\right| \leqslant \lambda^{-1} \ . \tag{1.5.47}$$

For a finite set (with multiplicity) $\Lambda \subseteq \mathbb{R}$, set

$$e_R[\Lambda] = \sum_{\lambda \in \Lambda} e_R(\lambda) \ . \tag{1.5.48}$$

Then

$$\begin{aligned} & \int_{R^{2-\varepsilon}}^{\infty} \left(\Theta_R^{I}(t) - \Theta_R^{*,I}(t)\right) \frac{dt}{t} \\ & = \sum_{j=0}^{2} \sum_p (-1)^{(j-1)(j-2)/2+p} p \left\{ e_R\left[\operatorname{Sp}\left(D_{Z_{j,R}}^{F,2,(p)}\right) \cap \,]0, R^{\kappa-2}[\right] \right. \\ & \qquad \left. - \, e_R\left[\operatorname{Sp}\left(D_{I_{j,R}}^{2,(p)}\right) \cap \,]0, R^{\kappa-2}[\right] \right\} \ . \end{aligned} \tag{1.5.49}$$

We will show that

$$e_R\left[\mathrm{Sp}\left(D^{F,2,(p)}_{Z_R}\right)\cap]0,R^{\kappa-2}[\right]-e_R\left[\mathrm{Sp}\left(D^{2,(p)}_{I_R}\right)\cap]0,R^{\kappa-2}[\right]=\mathscr{O}(R^{\kappa-1})\ . \tag{1.5.50}$$

The other terms can be estimated in the same way, and (1.5.45) follows.

Recall that Λ^p_R is defined in (1.3.147). By Theorem 1.3.19, we have

$$e_R\left[\mathrm{Sp}\left(D^{F,2,(p)}_{Z_R}\right)\cap]0,R^{\kappa-2}[\right]=\sum_{\rho\in\Lambda^p_R\,,\,0<|\rho|<R^{\kappa/2-1}}e_R(\rho^2)+\mathscr{O}(e^{-cR})\ . \tag{1.5.51}$$

Recall that $\Lambda^{*,p}_R$ is defined in (1.5.24). By (1.5.25), we have

$$e_R\left[\mathrm{Sp}\left(D^{2,(p)}_{I_R}\right)\cap]0,R^{\kappa-2}[\right]=\sum_{\lambda\in\Lambda^{*,p}_R\,,\,0<|\lambda|<R^{\kappa/2-1}}e_R(\lambda^2)\ . \tag{1.5.52}$$

By Appendix Proposition 1.8.3 and (1.5.47), we have

$$\sum_{\rho\in\Lambda^p_R\,,\,0<|\rho|<R^{-1+\kappa/2}}e_R(\rho^2)-\sum_{\lambda\in\Lambda^{*,p}_R\,,\,0<|\lambda|<R^{-1+\kappa/2}}e_R(\lambda^2)=\mathscr{O}(R^{\kappa-1})\ . \tag{1.5.53}$$

By (1.5.51), (1.5.52) and (1.5.53), we get (1.5.50). □

Theorem 1.5.8. *As $R\to\infty$, we have*

$$\sum_{j=0}^{2}(-1)^{(j-1)(j-2)/2}\Big({\zeta^L_{j,R}}'(0)-{\zeta^L_{*,j,R}}'(0)\Big)=\mathscr{O}(R^{\kappa-1})\ . \tag{1.5.54}$$

Proof. We combine Proposition 1.5.6, 1.5.7. □

Proof of Theorem 1.0.1 : We combine Proposition 1.5.4 and Theorem 1.5.5, 1.5.8. □

1.6. Asymptotics of the L^2-metrics on Mayer-Vietoris exact sequence.

In this section, we prove Theorem 1.0.2.

We use the notations and assumptions in §1.3.1 and §1.3.2.

In §1.6.1, we contruct a filtration of the Mayer-Vietoris exact sequence. More precisely, we extend the Mayer-Vietoris exact sequence to a commutative diagram with exact rows and columns. Moreover, we construct another commutative diagram (1.6.16), which is isomorphic to the original one. In §1.6.2, every object in diagram (1.6.16) is equipped with a metric (depending on R). We study the asymptotics of these metrics as $R\to\infty$. In §1.6.3, we study the asymptotics of the maps in diagram (1.6.16) as $R\to\infty$. In §1.6.4, with the help of diagram (1.6.16), we prove Theorem 1.0.2.

1.6.1. *A filtration of the Mayer-Vietoris exact sequence.*

Recall that (F,∇^F) is a flat vector bundle over Z, $Y\subseteq Z$ is a hypersurface cutting Z into Z_1, Z_2. For $R\geqslant 0$, we constructed $Z_{j,R}$ $(j=1,2)$ (resp. Z_R) by attaching a cylinder of length R (resp. $2R$) to Z_j (resp. Z). Then F extends to a flat vector bundle over Z_R.

The maps $\varphi_{j,R}:Z_{j,R}\to Z_j$ $(j=1,2)$ defined in (1.4.7) and $\varphi_R:Z_R\to Z$ defined in (1.3.32) are diffeomorphisms, which induce the following identifications

$$\varphi_{R*}:H^\bullet_{\mathbf{bd}}(Z_{j,R},F)\to H^\bullet_{\mathbf{bd}}(Z_j,F)\ ,\quad \varphi_{R*}:H^\bullet(Z_R,F)\to H^\bullet(Z,F)\ . \tag{1.6.1}$$

Since these diffeomorphisms commute with the injections $Z_j \hookrightarrow Z$ and $Z_{j,R} \hookrightarrow Z_R$, we get an isomorphism of long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^p_{\mathbf{bd}}(Z_{1,R},F) \arrow[r] \arrow[d, "\varphi_{R*}"] & H^p(Z_R,F) \arrow[r] \arrow[d, "\varphi_{R*}"] & H^p_{\mathbf{bd}}(Z_{2,R},F) \arrow[r] \arrow[d, "\varphi_{R*}"] & \cdots \\
\cdots \arrow[r] & H^p_{\mathbf{bd}}(Z_1,F) \arrow[r, "\alpha_p"] & H^p(Z,F) \arrow[r, "\beta_p"] & H^p_{\mathbf{bd}}(Z_2,F) \arrow[r, "\delta_p"] & \cdots ,
\end{tikzcd} \tag{1.6.2}$$

where each row is the classical Mayer-Vietoris exact sequence (1.0.16).

We recall that $D^F_{Z_{j,R}}$ $(j=1,2)$ (resp. $D^F_{Z_R}$) is the Hodge-de Rham operator (cf. (1.0.2)) acting on $\Omega^\bullet_{\mathbf{bd}}(Z_{j,R},F)$ (resp. $\Omega^\bullet(Z_R,F)$). Its kernel is denoted by $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,R},F)$ (resp. $\mathscr{H}^\bullet(Z_R,F)$). We recall that $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ $(j=1,2)$ is defined by (1.2.48) and $\mathscr{H}^\bullet(Z_{12,\infty},F)$ is defined by (1.3.10). We constructed in Definition 1.3.6, 1.4.4 the bijections

$$\begin{aligned} &\mathscr{F}_{Z_{j,R}} : \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F) \to \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,R},F) \ , \\ &\mathscr{F}_{Z_R} : \mathscr{H}^\bullet(Z_{12,\infty},F) \to \mathscr{H}^\bullet(Z_R,F) \ . \end{aligned} \tag{1.6.3}$$

By Theorem 1.1.1, $\mathscr{F}_{Z_{j,R}}$ and $\mathscr{F}_{Z_R}$ may be viewed as maps

$$\begin{aligned} &\mathscr{F}_{Z_{j,R}} : \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F) \to H^\bullet_{\mathbf{bd}}(Z_{j,R},F) \ , \\ &\mathscr{F}_{Z_R} : \mathscr{H}^\bullet(Z_{12,\infty},F) \to H^\bullet(Z_R,F) \ . \end{aligned} \tag{1.6.4}$$

Now we define the composition map

$$\begin{aligned} \widetilde{\mathscr{F}}_{Z_{j,R}} = \varphi_{R*} \circ \mathscr{F}_{Z_{j,R}} &: \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F) \to H^\bullet_{\mathbf{bd}}(Z_j,F) \ , \\ \widetilde{\mathscr{F}}_{Z_R} = \varphi_{R*} \circ \mathscr{F}_{Z_R} &: \mathscr{H}^\bullet(Z_{12,\infty},F) \to H^\bullet(Z,F) \ . \end{aligned} \tag{1.6.5}$$

We remark that these maps depend on R.

Recall that the inclusion $\mathscr{H}^\bullet_{L^2}(Z_{j,\infty},F) \subseteq \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ $(j=1,2)$ is defined in (1.2.49), and the inclusion $\mathscr{H}^\bullet_{L^2}(Z_{1,\infty},F) \oplus \mathscr{H}^\bullet_{L^2}(Z_{2,\infty},F) \subseteq \mathscr{H}^\bullet(Z_{12,\infty},F)$ is defined in (1.3.13). For simplicity, we denote $\mathscr{H}^\bullet_{L^2}(Z_{1,\infty},F) \oplus \mathscr{H}^\bullet_{L^2}(Z_{2,\infty},F) = \mathscr{H}^\bullet_{L^2}(Z_{12,\infty},F)$.

For R large enough, set

$$\begin{aligned} &K^\bullet_j = \widetilde{\mathscr{F}}_{Z_{j,R}}\big(\mathscr{H}^\bullet_{L^2}(Z_{j,\infty},F)\big) \subseteq H^\bullet_{\mathbf{bd}}(Z_j,F) \ , \quad \text{for } j=1,2 \ , \\ &K^\bullet_{12} = \widetilde{\mathscr{F}}_{Z_R}\big(\mathscr{H}^\bullet_{L^2}(Z_{12,\infty},F)\big) \subseteq H^\bullet(Z,F) \ . \end{aligned} \tag{1.6.6}$$

By Proposition 1.3.3 and Proposition 1.4.2, $K^\bullet_1$, $K^\bullet_2$ and $K^\bullet_{12}$ are independent of R. We define the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{H}^p_{L^2}(Z_{1,\infty},F) \arrow[r] \arrow[d, "\tilde{\mathscr{F}}_{Z_{1,R}}"] & \mathscr{H}^p_{L^2}(Z_{12,\infty},F) \arrow[r] \arrow[d, "\tilde{\mathscr{F}}_{Z_R}"] & \mathscr{H}^p_{L^2}(Z_{2,\infty},F) \arrow[r] \arrow[d, "\tilde{\mathscr{F}}_{Z_{2,R}}"] & 0 \\
0 \arrow[r] & K^p_1 \arrow[r, dashed] & K^p_{12} \arrow[r, dashed] & K^p_2 \arrow[r] & 0
\end{tikzcd} \tag{1.6.7}$$

where the first row consists of canonical injection/projection maps. By Proposition 1.3.3 and Proposition 1.4.2, diagram (1.6.7) is independent of R.

Set

$$L^\bullet_{j,\mathbf{bd}} = H^\bullet_{\mathbf{bd}}(Z_j,F)/K^\bullet_j \ , \quad L^\bullet_{12} = H^\bullet(Z,F)/K^\bullet_{12} \ . \tag{1.6.8}$$

Proposition 1.6.1. *We have the following commutative diagram with exact rows and columns*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & K_1^p \arrow[r] \arrow[d] & K_{12}^p \arrow[r] \arrow[d] & K_2^p \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^p_{\mathbf{bd}}(Z_1,F) \arrow[r, "\alpha_p"] \arrow[d] & H^p(Z,F) \arrow[r, "\beta_p"] \arrow[d] & H^p_{\mathbf{bd}}(Z_2,F) \arrow[r, "\delta_p"] \arrow[d] & \cdots \\
\cdots \arrow[r, dashed] & L^p_{1,\mathbf{bd}} \arrow[r, dashed, "\bar{\alpha}_p"] \arrow[d] & L^p_{12} \arrow[r, dashed, "\bar{\beta}_p"] \arrow[d] & L^p_{2,\mathbf{bd}} \arrow[r, dashed, "\bar{\delta}_p"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{1.6.9}
$$

where the maps $K_1^p \to K_{12}^p$ and $K_{12}^p \to K_2^p$ are defined by (1.6.7)*, the map $K_2^p \to K_1^{p+1}$ is zero, the second row is the classical Mayer-Vietoris exact sequence* (1.0.16)*, and the vertical maps are canonical injection/projection maps.*

Proof. We show that the upper left square commutes. It is equivalent to show that for any $\omega \in \mathscr{H}^p_{L^2}(Z_{1,\infty},F)$, we have

$$\alpha_p\big(\big[\mathscr{F}_{Z_{1,R}}(\omega,0)\big]\big) = [\mathscr{F}_{Z_R}(\omega,0,0)] \in H^p(Z,F)\ . \tag{1.6.10}$$

By (1.3.29) and (1.4.5), we have

$$F_{Z_R}(\omega,0,0)\big|_{Z_{1,R}} = F_{Z_{1,R}}(\omega,0)\ , \quad F_{Z_R}(\omega,0,0)\big|_{Z_{2,R}} = 0\ . \tag{1.6.11}$$

By (1.3.48) and (1.4.22), we have

$$\begin{aligned} \big[\mathscr{F}_{Z_{1,R}}(\omega,0)\big] &= \big[F_{Z_{1,R}}(\omega,0)\big] \in H^p_{\mathbf{bd}}(Z_1,F)\ ,\\ [\mathscr{F}_{Z_R}(\omega,0,0)] &= [F_{Z_R}(\omega,0,0)] \in H^p(Z,F)\ . \end{aligned} \tag{1.6.12}$$

By Proposition 1.1.2 and (1.6.11), we have

$$\alpha_p\big(\big[F_{Z_{1,R}}(\omega,0)\big]\big) = [F_{Z_R}(\omega,0,0)] \in H^p(Z,F)\ . \tag{1.6.13}$$

Then (1.6.10) follows from (1.6.12) and (1.6.13).

Proceeding in the same way, we can show that the upper right square commutes and $\delta_p\big(K_2^p\big) = 0$. We get the commutativity between the first and second rows.

The rests can be done by direct diagram chasing arguments. □

Let $\mathscr{L}_j^\bullet$ ($j=1,2$) be the set of limiting values (cf. (1.2.43)) of $\mathscr{H}^\bullet(Z_{j,\infty},F)$. Let $\mathscr{L}^\bullet_{j,\mathrm{abs/rel}}$ be the absolute/relative component (cf. (1.2.46)) of $\mathscr{L}_j^\bullet$. We still use the convention $\mathscr{L}^\bullet_{1,\mathbf{bd}} = \mathscr{L}^\bullet_{1,\mathrm{rel}}$ and $\mathscr{L}^\bullet_{2,\mathbf{bd}} = \mathscr{L}^\bullet_{1,\mathrm{abs}}$

We define, for $j=1,2$, the following commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{H}^p_{L^2}(Z_{j,\infty},F) \arrow[r] \arrow[d, "\tilde{\mathscr{F}}_{Z_{j,R}}"] & \mathscr{H}^p_{\mathbf{bd}}(Z_{j,\infty},F) \arrow[r] \arrow[d, "\tilde{\mathscr{F}}_{Z_{j,R}}"] & \mathscr{L}^p_{j,\mathbf{bd}} \arrow[r] \arrow[d, dashed, "\overline{\mathscr{F}}_{Z_{j,R}}"] & 0 \\
0 \arrow[r] & K_j^p \arrow[r] & H^p_{\mathbf{bd}}(Z_j,F) \arrow[r] & L^p_{j,\mathbf{bd}} \arrow[r] & 0
\end{tikzcd}
\tag{1.6.14}
$$

where the first row is defined by (1.2.49), the second row consists of canonical injection/projection maps. We define the following commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{H}^p_{L^2}(Z_{12,\infty},F) \arrow[r] \arrow[d, "\widetilde{\mathscr{F}}_{Z_R}"] & \mathscr{H}^p(Z_{12,\infty},F) \arrow[r] \arrow[d, "\widetilde{\mathscr{F}}_{Z_R}"] & \mathscr{L}_1^p \cap \mathscr{L}_2^p \arrow[r] \arrow[d, dashed, "\overline{\mathscr{F}}_{Z_R}"] & 0 \\
0 \arrow[r] & K^p_{12} \arrow[r] & H^p(Z,F) \arrow[r] & L^p_{12} \arrow[r] & 0
\end{tikzcd}
\tag{1.6.15}
$$

where the first row is defined by (1.3.13), the second row consists of canonical injection/projection maps.

By (1.6.9), (1.6.14) and (1.6.15), we get the following commutative diagram with exact rows and columns, which is the analytic counterpart of (1.6.9),

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathscr{H}^p_{L^2}(Z_{1,\infty},F) \arrow[r] \arrow[d] & \mathscr{H}^p_{L^2}(Z_{12,\infty},F) \arrow[r] \arrow[d] & \mathscr{H}^p_{L^2}(Z_{2,\infty},F) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty},F) \arrow[r, "\alpha_p(R)"] \arrow[d] & \mathscr{H}^p(Z_{12,\infty},F) \arrow[r, "\beta_p(R)"] \arrow[d] & \mathscr{H}^p_{\mathbf{bd}}(Z_{2,\infty},F) \arrow[r, "\delta_p(R)"] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathscr{L}^p_{1,\mathbf{bd}} \arrow[r, "\bar{\alpha}_p(R)"] \arrow[d] & \mathscr{L}_1^p \cap \mathscr{L}_2^p \arrow[r, "\bar{\beta}_p(R)"] \arrow[d] & \mathscr{L}^p_{2,\mathbf{bd}} \arrow[r, "\bar{\delta}_p(R)"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{1.6.16}
$$

where the first row consists of canonical injection/projection maps, the columns are defined by (1.2.49) and (1.3.13), and

$$
\begin{aligned}
\alpha_p(R) &= \left(\widetilde{\mathscr{F}}_{Z_R}\right)^{-1} \circ \alpha_p \circ \widetilde{\mathscr{F}}_{Z_{1,R}} \,, & \overline{\alpha}_p(R) &= \left(\overline{\mathscr{F}}_{Z_R}\right)^{-1} \circ \overline{\alpha}_p \circ \overline{\mathscr{F}}_{Z_{1,R}} \,,\\
\beta_p(R) &= \left(\widetilde{\mathscr{F}}_{Z_{2,R}}\right)^{-1} \circ \beta_p \circ \widetilde{\mathscr{F}}_{Z_R} \,, & \overline{\beta}_p(R) &= \left(\overline{\mathscr{F}}_{Z_{2,R}}\right)^{-1} \circ \overline{\beta}_p \circ \overline{\mathscr{F}}_{Z_R} \,,\\
\delta_p(R) &= \left(\widetilde{\mathscr{F}}_{Z_{1,R}}\right)^{-1} \circ \delta_p \circ \widetilde{\mathscr{F}}_{Z_{2,R}} \,, & \overline{\delta}_p(R) &= \left(\overline{\mathscr{F}}_{Z_{1,R}}\right)^{-1} \circ \overline{\delta}_p \circ \overline{\mathscr{F}}_{Z_{2,R}} \,.
\end{aligned}
\tag{1.6.17}
$$

1.6.2. *Asymptotics of the L^2-metrics.*

We begin by equipping the spaces in the second row of diagram (1.6.16) with metrics.

We recall that the metric $\|\cdot\|_{\mathscr{H}^\bullet(Z_{12,\infty},F),R}$ on $\mathscr{H}^\bullet(Z_{12,\infty},F)$ is defined by (1.3.14). Let $\mathscr{F}^*_{Z_R}\left(\|\cdot\|_{H^\bullet(Z_R,F)}\right)$ be another metric on $\mathscr{H}^\bullet(Z_{12,\infty},F)$, which is the pull-back of the L^2-metric (defined in §1.0.4) $\|\cdot\|_{H^\bullet(Z_R,F)}$ on $H^\bullet(Z_R,F)$ via $\mathscr{F}_{Z_R}$ (cf. Definition 1.3.6).

We recall that the metric $\|\cdot\|_{\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F),R}$ ($j=1,2$) on $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ is defined by (1.4.19). Let $\mathscr{F}^*_{Z_{j,R}}\left(\|\cdot\|_{H^\bullet_{\mathbf{bd}}(Z_{j,R},F)}\right)$ be another metric on $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$, which is the pull-back of the L^2-metric $\|\cdot\|_{H^\bullet_{\mathbf{bd}}(Z_{j,R},F)}$ on $H^\bullet_{\mathbf{bd}}(Z_{j,R},F)$ via $\mathscr{F}_{Z_{j,R}}$ (cf. Definition 1.4.4).

Proposition 1.6.2. *There exists $c>0$ such that as $R\to+\infty$, we have*

$$
\begin{aligned}
\mathscr{F}^*_{Z_{j,R}}\left(\|\cdot\|_{H^\bullet_{\mathbf{bd}}(Z_{j,R},F)}\right) &= \|\cdot\|_{\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F),R} + \mathscr{O}(e^{-cR}) \,, \quad \textit{for } j=1,2 \,,\\
\mathscr{F}^*_{Z_R}\left(\|\cdot\|_{H^\bullet(Z_R,R)}\right) &= \|\cdot\|_{\mathscr{H}^\bullet(Z_{12,\infty},F),R} + \mathscr{O}(e^{-cR}) \,.
\end{aligned}
\tag{1.6.18}
$$

Proof. The first identitiy is a direct consequence of Propostion 1.4.3, 1.4.5. The second identity is a direct consequence of Proposition 1.3.4, 1.3.7. □

Now we equip the spaces in the third row of diagram (1.6.16) with metrics.

Let $\|\cdot\|_{\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet,R}$ be the quotient metric on $\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet$ induced by $\|\cdot\|_{\mathscr{H}^\bullet(Z_{12,\infty},F),R}$ via the vertical map $\mathscr{H}^\bullet(Z_{12,\infty},F)\to\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet$ in diagram (1.6.16). Let $\|\cdot\|_{\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet}$ be another metric on $\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet$, which is induced by the L^2-metric $\|\cdot\|_Y$ on $\mathscr{H}^\bullet(Y,F[du])$ via the inclusion $\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet\subseteq\mathscr{H}^\bullet(Y,F[du])$ (cf. (1.2.43)).

Proceeding in the same way, we define metrics $\|\cdot\|_{\mathscr{L}_{j,\mathbf{bd}}^\bullet,R}$ and $\|\cdot\|_{\mathscr{L}_{j,\mathbf{bd}}^\bullet}$ on $\mathscr{L}_{j,\mathbf{bd}}^\bullet$.

Proposition 1.6.3. *As $R\to+\infty$, we have*

$$\begin{aligned}\|\cdot\|^2_{\mathscr{L}_{j,\mathbf{bd}}^\bullet,R}&=R\,\|\cdot\|^2_{\mathscr{L}_{j,\mathbf{bd}}^\bullet}+\mathscr{O}(1)\;,\quad \textit{for } j=1,2\;,\\ \|\cdot\|^2_{\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet,R}&=2R\,\|\cdot\|^2_{\mathscr{L}_1^\bullet\cap\mathscr{L}_2^\bullet}+\mathscr{O}(1)\;.\end{aligned}\tag{1.6.19}$$

Proof. We only prove the first one for $j=2$. The others can be proved in the same way.

We recall that $\mathscr{H}_{\mathbf{bd}}^\bullet(Z_{2,\infty},F)$ is defined by (1.2.48). By the definition of quotient metric, for any $\hat\omega\in\mathscr{L}_{2,\mathbf{bd}}^\bullet$, we have

$$\|\hat\omega\|^2_{\mathscr{L}_{2,\mathbf{bd}}^\bullet,R}=\inf_{(\omega,\hat\omega)\in\mathscr{H}_{\mathbf{bd}}^\bullet(Z_{2,\infty},F)}\|(\omega,\hat\omega)\|^2_{\mathscr{H}_{\mathbf{bd}}^\bullet(Z_{2,\infty},F),R}\;.\tag{1.6.20}$$

We recall that $I_{2,\infty}Y\subseteq Z_{2,\infty}$ is its cylinder part, defined in §1.3.1. On $I_{2,\infty}Y$, let $\omega=\omega^{\mathrm{zm}}+\omega^{\mathrm{nz}}$ be the decomposition of ω into zero-mode and non zero-mode parts, defined in (1.2.16). Recall that $\pi_Y: I_{2,\infty}Y\to Y$ is the natural projection. We have $\pi_Y^*\hat\omega=\omega^{\mathrm{zm}}$. As a consequence, we have

$$\|\omega^{\mathrm{zm}}\|^2_{I_{2,R}Y}=R\,\|\hat\omega\|^2_Y=R\,\|\hat\omega\|^2_{\mathscr{L}_{2,\mathbf{bd}}^\bullet}\;,\tag{1.6.21}$$

where $I_{2,R}Y\subseteq Z_{2,R}$ is the cylinder part of $Z_{2,R}$, also defined in §1.3.1. Thus

$$\begin{aligned}\|(\omega,\hat\omega)\|^2_{\mathscr{H}_{\mathbf{bd}}^\bullet(Z_{2,\infty},F),R}-R\,\|\hat\omega\|^2_{\mathscr{L}_{2,\mathbf{bd}}^\bullet}&=\|\omega\|^2_{Z_{2,R}}-\|\omega^{\mathrm{zm}}\|^2_{I_{2,R}Y}\\&=\|\omega\|^2_{Z_{2,0}}+\|\omega^{\mathrm{nz}}\|^2_{I_{2,R}Y}\;.\end{aligned}\tag{1.6.22}$$

In particular, we have

$$\|(\omega,\hat\omega)\|^2_{\mathscr{H}_{\mathbf{bd}}^\bullet(Z_{2,\infty},F),R}\geqslant R\,\|\hat\omega\|^2_{\mathscr{L}_{2,\mathbf{bd}}^\bullet}\;.\tag{1.6.23}$$

By (1.6.20), (1.6.22) and (1.6.23), it is sufficient to show that there exists $C>0$ such that for any $\hat\omega\in\mathscr{L}_{2,\mathbf{bd}}^\bullet$, there exists $(\omega,\hat\omega)\in\mathscr{H}_{\mathbf{bd}}^\bullet(Z_{2,\infty},F)$ such that for any $R>0$,

$$\|\omega\|^2_{Z_{2,0}}+\|\omega^{\mathrm{nz}}\|^2_{I_{2,R}Y}\leqslant C\,\|\hat\omega\|^2_{\mathscr{L}_{2,\mathbf{bd}}^\bullet}\;.\tag{1.6.24}$$

In the rest of the proof, we choose ω a generalized eigensection of $D^F_{Z_{2,\infty}}$ associated with $\lambda=0$ such that $(\omega,\hat\omega)\in\mathscr{H}_{\mathbf{bd}}^\bullet(Z_{2,\infty},F)$. The existence and uniqueness of ω comes from Remark 1.2.7. By (1.2.33), there exists $C_1>0$ such that for any $\hat\omega$ and its associated generalized eigensection ω, we have

$$\|\omega\|^2_{Z_{2,0}}\leqslant C_1\|\hat\omega\|^2_Y=C_1\|\hat\omega\|^2_{\mathscr{L}_{2,\mathbf{bd}}^\bullet}\;.\tag{1.6.25}$$

Applying Lemma 1.2.1 and (1.3.43) with $Z_{1,0}$ replaced by $Z_{2,0}$, there exists $C_2>0$ such that for any generalized eigensection ω associated with $\lambda=0$, we have

$$\|\omega^{\mathrm{nz}}\|^2_{I_{2,R}Y}\leqslant\|\omega^{\mathrm{nz}}\|^2_{I_{2,\infty}Y}\leqslant C_2\|\omega\|^2_{Z_{2,0}}\;.\tag{1.6.26}$$

By (1.6.25)-(1.6.26), we get (1.6.24). □

1.6.3. *Asymptotics of the horizontal maps.*

First we consider the second row of diagram (1.6.16).

We recall that the operators $du\wedge$, $i_{\frac{\partial}{\partial u}}$ and $c(\frac{\partial}{\partial u})$ on $\Omega^\bullet(Y,F[du])$ or $\mathscr{H}^\bullet(Y,F[du])$ are defined in (1.2.4).

In the sequel, by $\mathscr{O}\left(e^{-cR}\right)$, we mean a number bounded by Ce^{-cR} with $c,C>0$ uniquely determined by Z_1, Z_2, F. We will use the notations $\mathscr{O}\left(R^{-1}\right)$, $\mathscr{O}\left(R^{-2}\right)$, etc., in the same way.

Proposition 1.6.4. *For $(\omega,\hat\omega)\in\mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty},F)$ and $(\mu_1,\mu_2,\hat\mu)\in\mathscr{H}^p(Z_{12,\infty},F)$, we have*

$$\begin{aligned}&\left\langle\alpha_p(R)(\omega,\hat\omega),(\mu_1,\mu_2,\hat\mu)\right\rangle_{\mathscr{H}^p(Z_{12,\infty},F),R}\\ &=\ \langle\omega,\mu_1\rangle_{Z_{1,R}}+\mathscr{O}\left(e^{-cR}\right)\|(\omega,\hat\omega)\|_{\mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty},F)}\|(\mu_1,\mu_2,\hat\mu)\|_{\mathscr{H}^p(Z_{12,\infty},F)}\ .\end{aligned}\tag{1.6.27}$$

For $(\omega_1,\omega_2,\hat\omega)\in\mathscr{H}^p(Z_{12,\infty},F)$ and $(\mu,\hat\mu)\in\mathscr{H}^p_{\mathbf{bd}}(Z_{2,\infty},F)$, we have

$$\begin{aligned}&\left\langle\beta_p(R)(\omega_1,\omega_2,\hat\omega),(\mu,\hat\mu)\right\rangle_{\mathscr{H}^p_{\mathbf{bd}}(Z_{2,\infty},F),R}\\ &=\ \langle\omega_2,\mu\rangle_{Z_{2,R}}+\mathscr{O}\left(e^{-cR}\right)\|(\omega_1,\omega_2,\hat\omega)\|_{\mathscr{H}^p(Z_{12,\infty},F)}\|(\mu,\hat\mu)\|_{\mathscr{H}^p_{\mathbf{bd}}(Z_{2,\infty},F)}\ .\end{aligned}\tag{1.6.28}$$

For $(\omega,\hat\omega)\in\mathscr{H}^p_{\mathbf{bd}}(Z_{2,\infty},F)$ and $(\mu,\hat\mu)\in\mathscr{H}^{p+1}_{\mathbf{bd}}(Z_{1,\infty},F)$, we have

$$\begin{aligned}&\left\langle\delta_p(R)(\omega,\hat\omega),(\mu,\hat\mu)\right\rangle_{\mathscr{H}^{p+1}(Z_{1,\infty},F),R}\\ &=\ \left\langle\hat\omega,i_{\frac{\partial}{\partial u}}\hat\mu\right\rangle_Y+\mathscr{O}\left(e^{-cR}\right)\|(\omega,\hat\omega)\|_{\mathscr{H}^p_{\mathbf{bd}}(Z_{2,\infty},F)}\|(\mu,\hat\mu)\|_{\mathscr{H}^{p+1}_{\mathbf{bd}}(Z_{1,\infty},F)}\ .\end{aligned}\tag{1.6.29}$$

Proof. Once again, we recall that $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ $(j=1,2)$ is defined by (1.2.48) and $\mathscr{H}^\bullet(Z_{12,\infty},F)$ is defined by (1.3.10).

For $(\omega,\hat\omega)\in\mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty},F)$, we denote

$$\alpha_p(R)(\omega,\hat\omega)=(\omega_1',\omega_2',\hat\omega')\in\mathscr{H}^p(Z_{12,\infty},F)\ .\tag{1.6.30}$$

By (1.6.17) and (1.6.30), we have

$$\alpha_p\left(\left[\mathscr{F}_{Z_{1,R}}(\omega,\hat\omega)\right]\right)=\left[\mathscr{F}_{Z_R}(\omega_1',\omega_2',\hat\omega')\right]\in H^p(Z,F)\ .\tag{1.6.31}$$

Then, by Proposition 1.1.3, for $(\mu_1,\mu_2,\hat\mu)\in\mathscr{H}^p(Z_{12,\infty},F)$, we have

$$\left\langle\mathscr{F}_{Z_R}(\omega_1',\omega_2',\hat\omega'),\mathscr{F}_{Z_R}(\mu_1,\mu_2,\hat\mu)\right\rangle_{Z_R}=\left\langle\mathscr{F}_{Z_{1,R}}(\omega,\hat\omega),\mathscr{F}_{Z_R}(\mu_1,\mu_2,\hat\mu)\right\rangle_{Z_{1,R}}\ .\tag{1.6.32}$$

By Proposition 1.6.2, we have

$$\begin{aligned}&\left\langle(\omega_1',\omega_2',\hat\omega'),(\mu_1,\mu_2,\hat\mu)\right\rangle_{\mathscr{H}^p(Z_{12,\infty},F),R}\\ &=\left\langle\mathscr{F}_{Z_R}(\omega_1',\omega_2',\hat\omega'),\mathscr{F}_{Z_R}(\mu_1,\mu_2,\hat\mu)\right\rangle_{Z_R}\left(1+\mathscr{O}\left(e^{-cR}\right)\right)\ .\end{aligned}\tag{1.6.33}$$

By Proposition 1.3.4, 1.3.7, 1.4.3, 1.4.5, we have

$$\begin{aligned}&\left\langle\mathscr{F}_{Z_{1,R}}(\omega,\hat\omega),\mathscr{F}_{Z_R}(\mu_1,\mu_2,\hat\mu)\right\rangle_{Z_{1,R}}\\ &=\ \langle\omega,\mu_1\rangle_{Z_{1,R}}+\mathscr{O}\left(e^{-cR}\right)\|(\omega,\hat\omega)\|_{\mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty},F)}\|(\mu_1,\mu_2,\hat\mu)\|_{\mathscr{H}^p(Z_{12,\infty},F)}\ .\end{aligned}\tag{1.6.34}$$

By (1.6.30) and (1.6.32)-(1.6.34), we get (1.6.27).

The second and third identities can be proved following in the same way. □

Now we consider the third row of diagram (1.6.16). We remark that the exact sequence (1.5.11) involves the same vector spaces appearing in the third row of diagram (1.6.16).

Proposition 1.6.5. *As $R \to \infty$, we have*

$$\begin{aligned} \bar{\alpha}_p(R) &= \frac{1}{2}\alpha_{p,\mathscr{L}} + \mathscr{O}\left(R^{-1}\right) \ , \\ \bar{\beta}_p(R) &= \beta_{p,\mathscr{L}} + \mathscr{O}\left(R^{-1}\right) \ , \\ \bar{\delta}_p(R) &= R^{-1}\delta_{p,\mathscr{L}} + \mathscr{O}\left(R^{-2}\right) \ . \end{aligned} \tag{1.6.35}$$

Proof. We only prove the first one. The rests can be proved in the same way.

By Remark 1.2.7, for $\hat{\omega} \in \mathscr{L}^p_{1,\mathbf{bd}}$, there exists $(\omega, \hat{\omega}) \in \mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty}, F)$ such that ω is a generalized eigensection. We denote

$$\alpha_p(R)(\omega, \hat{\omega}) = (\omega_1', \omega_2', \hat{\omega}') \in \mathscr{H}^p(Z_{12,\infty}, F) \ . \tag{1.6.36}$$

Then, by (1.6.17),

$$\bar{\alpha}_p(R)(\hat{\omega}) = \hat{\omega}' \ . \tag{1.6.37}$$

We need to show that

$$\left\| \hat{\omega}' - \frac{1}{2}\alpha_{p,\mathscr{L}}(\hat{\omega}) \right\|^2_{\mathscr{L}^p_1 \cap \mathscr{L}^p_2} = \mathscr{O}\left(R^{-2}\right) \|\hat{\omega}\|^2_Y \ . \tag{1.6.38}$$

By Proposition 1.6.3, it is sufficient to show that

$$\left\| \hat{\omega}' - \frac{1}{2}\alpha_{p,\mathscr{L}}(\hat{\omega}) \right\|^2_{\mathscr{L}^p_1 \cap \mathscr{L}^p_2, R} = \mathscr{O}\left(R^{-1}\right) \|\hat{\omega}\|^2_Y \ . \tag{1.6.39}$$

By Remark 1.2.7, there exists $(\omega_1'', \omega_2'', \hat{\omega}'') \in \mathscr{H}^p(Z_{12,\infty}, F)$ such that ω_1'' and ω_2'' are generalized eigensections and

$$\hat{\omega}'' = \frac{1}{2}\alpha_{p,\mathscr{L}}(\hat{\omega}) \ . \tag{1.6.40}$$

Since $\|\cdot\|_{\mathscr{L}^p_1 \cap \mathscr{L}^p_2, R}$ is the quotient metric induced by $\|\cdot\|_{\mathscr{H}^p(Z_{12,\infty}, F), R}$, for proving (1.6.39), it is sufficient to show that

$$\left\| (\omega_1', \omega_2', \hat{\omega}') - (\omega_1'', \omega_2'', \hat{\omega}'') \right\|^2_{\mathscr{H}^p(Z_{12,\infty}, F), R} = \mathscr{O}\left(R^{-1}\right) \|\hat{\omega}\|^2_Y \ . \tag{1.6.41}$$

By Riesz representation theorem, it is equivalent to show that for any $(\mu_1, \mu_2, \hat{\mu}) \in \mathscr{H}^p(Z_{12,\infty}, F)$, we have

$$\begin{aligned} &\left\langle (\omega_1', \omega_2', \hat{\omega}') - (\omega_1'', \omega_2'', \hat{\omega}''), (\mu_1, \mu_2, \hat{\mu}) \right\rangle_{\mathscr{H}^p(Z_{12,\infty}, F), R} \\ &= \mathscr{O}\left(R^{-1/2}\right) \left\|\hat{\omega}\right\|_Y \left\|(\mu_1, \mu_2, \hat{\mu})\right\|_{\mathscr{H}^p(Z_{12,\infty}, F), R} . \end{aligned} \tag{1.6.42}$$

By Proposition 1.6.4 and (1.6.36), we have

$$\begin{aligned} &\left\langle (\omega_1', \omega_2', \hat{\omega}'), (\mu_1, \mu_2, \hat{\mu}) \right\rangle_{\mathscr{H}^p(Z_{12,\infty}, F), R} \\ &= \ \langle \omega, \mu_1 \rangle_{Z_{1,R}} + \mathscr{O}\left(e^{-cR}\right) \left\|(\omega, \hat{\omega})\right\|_{\mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty}, F)} \left\|(\mu_1, \mu_2, \hat{\mu})\right\|_{\mathscr{H}^p(Z_{12,\infty}, F)} . \end{aligned} \tag{1.6.43}$$

Since ω is a generalized eigensection, by (1.2.33), we have

$$\left\|(\omega, \hat{\omega})\right\|_{\mathscr{H}^p_{\mathbf{bd}}(Z_{1,\infty}, F)} = \|\omega\|_{Z_{1,0}} = \mathscr{O}(1) \|\hat{\omega}\|_Y \ . \tag{1.6.44}$$

By (1.6.43) and (1.6.44), we get

$$\begin{aligned} &\left\langle (\omega_1', \omega_2', \hat{\omega}'), (\mu_1, \mu_2, \hat{\mu}) \right\rangle_{\mathscr{H}^p(Z_{12,\infty}, F), R} \\ &= \langle \omega, \mu_1 \rangle_{Z_{1,R}} + \mathscr{O}\left(e^{-cR}\right) \left\|\hat{\omega}\right\|_Y \left\|(\mu_1, \mu_2, \hat{\mu})\right\|_{\mathscr{H}^p(Z_{12,\infty}, F)} . \end{aligned} \tag{1.6.45}$$

The following identity is just the definition of $\langle\cdot,\cdot\rangle_{\mathscr{H}^p(Z_{12,\infty},F),R}$ (cf. (1.3.14)),

$$\Big\langle(\omega_1'',\omega_2'',\hat{\omega}''),(\mu_1,\mu_2,\hat{\mu})\Big\rangle_{\mathscr{H}^p(Z_{12,\infty},F),R}=\langle\omega_1'',\mu_1\rangle_{Z_{1,R}}+\langle\omega_2'',\mu_2\rangle_{Z_{2,R}}\;. \tag{1.6.46}$$

Comparing (1.3.15), (1.6.42), (1.6.45) and (1.6.46), it remains to show that

$$\begin{aligned}&\langle\omega,\mu_1\rangle_{Z_{1,R}}-\langle\omega_1'',\mu_1\rangle_{Z_{1,R}}-\langle\omega_2'',\mu_2\rangle_{Z_{2,R}}\\&=\mathscr{O}\left(R^{-1/2}\right)\|\hat{\omega}\|_Y\,\|(\mu_1,\mu_2,\hat{\mu})\|_{\mathscr{H}^p(Z_{12,\infty},F),R}\;.\end{aligned} \tag{1.6.47}$$

Since ω_1'', ω_2'' and ω are generalized eigensections, by using Lemma 1.2.1 and (1.2.33) in the same way as in the proof of Proposition 1.6.3, we get

$$\begin{aligned}&\big\langle\omega_j'',\mu_j\big\rangle_{Z_{j,R}}\\&=R\langle\hat{\omega}'',\hat{\mu}\rangle_Y+\mathscr{O}\big(1\big)\|\hat{\omega}\|_Y\|\hat{\mu}\|_Y\\&=R\langle\hat{\omega}'',\hat{\mu}\rangle_Y+\mathscr{O}\left(R^{-1/2}\right)\|\hat{\omega}\|_Y\big\|(\mu_1,\mu_2,\hat{\mu})\big\|_{\mathscr{H}^p(Z_{12,\infty},F),R}\;,\qquad\text{for } j=1,2\;,\\&\langle\omega,\mu_1\rangle_{Z_{1,R}}\\&=R\langle\hat{\omega},\hat{\mu}\rangle_Y+\mathscr{O}\big(1\big)\|\hat{\omega}\|_Y\|\hat{\mu}\|_Y\\&=R\langle\hat{\omega},\hat{\mu}\rangle_Y+\mathscr{O}\left(R^{-1/2}\right)\|\hat{\omega}\|_Y\big\|(\mu_1,\mu_2,\hat{\mu})\big\|_{\mathscr{H}^p(Z_{12,\infty},F),R}\;.\end{aligned} \tag{1.6.48}$$

By (1.5.8) and (1.6.40), we have

$$\langle\hat{\omega}'',\hat{\mu}\rangle_Y=\frac{1}{2}\big\langle\alpha_{p,\mathscr{L}}(\hat{\omega}),\hat{\mu}\big\rangle_Y=\frac{1}{2}\langle\hat{\omega},\hat{\mu}\rangle_Y\;. \tag{1.6.49}$$

By (1.6.48) and (1.6.49), we obtain (1.6.47). This finishes the proof of the first equation. □

Remark 1.6.6. A special case of the problem addressed in this subsection was considered by Müller-Strohmaier [MS10]. Considering the following Mayer-Vietoris exact sequence

$$\cdots\longrightarrow H^p_{\mathrm{rel}}(Z_{1,R},\mathbb{C})\xrightarrow{\alpha_p}H^p_{\mathrm{abs}}(Z_{1,R},\mathbb{C})\xrightarrow{\beta_p}H^p(Y,\mathbb{C})\xrightarrow{\delta_p}\cdots\;, \tag{1.6.50}$$

they gave an asymptotic estimate of the sesquilinear form

$$H^p(Y,\mathbb{C})\times H^p(Y,\mathbb{C})\to\mathbb{C}\;;\;(\phi,\varphi)\mapsto\big\langle\delta_p\phi,\delta_p\varphi\big\rangle\;, \tag{1.6.51}$$

as $R\to\infty$ ([MS10, Theorem 3.3]), where $\langle\cdot,\cdot\rangle$ is the L^2-metric on $H^\bullet_{\mathrm{rel}}(Z_{1,R},\mathbb{C})$.

1.6.4. *Torsion of the Mayer-Vietoris exact sequence : proof of Theorem 1.0.2.*

First we state a technical lemma.

For $A:V\to W$ a linear map between Hermitian vector spaces of the same dimension, we denote by $\det(A)$ the determinant of the matrix of A under any orthogonal bases, which is well-defined up to $\mathrm{U}(1):=\big\{z\in\mathbb{C}\;:\;|z|=1\big\}$.

We recall that $\det^*(\cdot)$ is defined by (1.0.25).

Lemma 1.6.7. *Let V be a Hermitian vector space, $H_1,H_2\subseteq V$ two vector subspaces. Let P_j be the orthogonal projection to H_j for $j=1,2$. We have*

$$\begin{aligned}&\big|\det(P_1|_{\mathrm{Im}(P_2P_1)})\big|=\big|\det(P_2|_{\mathrm{Im}(P_1P_2)})\big|\\&=\det{}^*\big(\mathrm{Id}-P_1-P_2+P_1P_2+P_2P_1\big)^{\frac{1}{4}}\;.\end{aligned} \tag{1.6.52}$$

Proof. We claim that there exists an orthogonal decomposition $V = \bigoplus_k V_k$ such that $\dim V_k \leqslant 2$ and $H_j = \bigoplus_k (V_k \cap H_j)$ for $j = 1, 2$. Once the claim is proved, we may suppose that $\dim V \leqslant 2$. Then the only non trivial case is $\dim V = 2$ and $\dim H_1 = \dim H_2 = 1$. We may suppose that

$$V = \mathbb{C}^2 \,, \quad H_1 = \mathbb{C}(1,0) \,, \quad H_2 = \mathbb{C}(\cos\theta, \sin\theta) \,, \quad \text{with } 0 \leqslant \theta \leqslant \frac{\pi}{2} \,. \tag{1.6.53}$$

We have $\left|\det(P_1|_{\mathrm{Im}(P_2P_1)})\right| = \left|\det(P_2|_{\mathrm{Im}(P_1P_2)})\right| = \cos\theta$, and

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \,, \quad P_2 = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \,. \tag{1.6.54}$$

Then (1.6.52) follows from a direct calculation.

Now we prove the claim. The operator $P_1P_2P_1$ (resp. $P_2P_1P_2$) acting on H_1 (resp. H_2) is self-adjoint, let

$$H_1 = \bigoplus_{0\leqslant\lambda\leqslant 1} H_1^\lambda \,, \quad H_2 = \bigoplus_{0\leqslant\lambda\leqslant 1} H_2^\lambda \tag{1.6.55}$$

be the associated spectral deompositions, i.e.,

$$P_1P_2P_1\big|_{H_1^\lambda} = \lambda \mathrm{Id} \,, \quad P_2P_1P_2\big|_{H_2^\lambda} = \lambda \mathrm{Id} \,. \tag{1.6.56}$$

We have

$$H_1^1 = H_2^1 = H_1 \cap H_2 \,, \quad H_1^0 = H_1 \cap H_2^\perp \,, \quad H_2^0 = H_2 \cap H_1^\perp \,. \tag{1.6.57}$$

We get the orthogonal decomposition

$$V = (H_1 + H_2)^\perp \oplus (H_1 \cap H_2) \oplus (H_1 \cap H_2^\perp) \oplus (H_2 \cap H_1^\perp) \oplus \bigoplus_{0<\lambda<1} (H_1^\lambda + H_2^\lambda) \,, \tag{1.6.58}$$

which is invariant under the actions of P_1 and P_2. The problem decomposes to each block. In $H_1 \cap H_2$, the vector spaces in question are both the whole space. We take $(e_j)_j$ an orthogonal basis of $H_1 \cap H_2$ and choose $V_j = \mathbb{C}e_j$. For similar reasons, the claim is true for $(H_1 + H_2)^\perp$, $H_1 \cap H_2^\perp$ and $H_2 \cap H_1^\perp$. For $H_1^\lambda + H_2^\lambda$ with $0 < \lambda < 1$, let $(v_j)_{1\leqslant j\leqslant r}$ be an orthogonal basis of H_1^λ, let V_j be the vector subspace spanned by $\{v_j, P_2v_j\}$. These V_j satisfy the desired condition. □

We briefly recall some properties of torsion (cf. [BGS88a, §1a]), which are of constant use in this subsection. For a finite acyclic complex $(V^\bullet, \partial)$ of Hermitian vector spaces, we denote by $\mathscr{T}(V^\bullet, \partial)$ its torsion (cf. (1.0.15)).

- Let $(V^\bullet[n], \partial)$ be the n-th right-shift of $(V^\bullet, \partial)$, i.e., $V^k[n] = V^{k-n}$, then

$$\mathscr{T}(V^\bullet[n], \partial) = \left(\mathscr{T}(V^\bullet, \partial)\right)^{(-1)^n} \,. \tag{1.6.59}$$

- If $(V^\bullet, \partial)$ is the direct sum of two complexes $(V_1^\bullet, \partial_1)$ and $(V_2^\bullet, \partial_2)$, then

$$\mathscr{T}(V^\bullet, \partial) = \mathscr{T}(V_1^\bullet, \partial_1) \cdot \mathscr{T}(V_2^\bullet, \partial_2) \,. \tag{1.6.60}$$

- For a short acyclic complex

$$(V^\bullet, \partial) : 0 \to V^1 \to V^2 \to 0 \,, \tag{1.6.61}$$

let A be the matrix of $\partial : V^1 \to V^2$ with respect to any orthogonal bases, then

$$\mathscr{T}(V^\bullet, \partial) = |\det(A)| \,. \tag{1.6.62}$$

Let $T_{\mathscr{L}}$ be the torsion of the exact sequence (1.5.11) equipped with metrics $\|\cdot\|_{\mathscr{L}^\bullet_{j,\mathbf{bd}}}$ $(j=1,2)$ and $\|\cdot\|_{\mathscr{L}^\bullet_1\cap\mathscr{L}^\bullet_2}$. We calculate $T_{\mathscr{L}}$ in the follows.

We recall that $\mathscr{L}^\bullet_{j,\mathrm{abs}} \subseteq \mathscr{H}^\bullet(Y,F)$ $(j=1,2)$ is the absolute component of $\mathscr{L}^\bullet_j \subseteq \mathscr{H}^\bullet(Y,F[du])$, defined by (1.2.46). Let $\mathscr{L}^{\bullet,\perp}_{j,\mathrm{abs}} \subseteq \mathscr{H}^\bullet(Y,F)$ be its orthogonal complement with respect to the L^2-metric on $\mathscr{H}^\bullet(Y,F)$. We define $S^p_j \in \mathrm{End}\big(\mathscr{H}^p(Y,F)\big)$ as follows

$$S^p_j = \mathrm{Id}_{\mathscr{L}^p_{j,\mathrm{abs}}} - \mathrm{Id}_{\mathscr{L}^{p,\perp}_{j,\mathrm{abs}}}\ . \tag{1.6.63}$$

By identifying $\mathscr{H}^p(Y,F)$ to $\mathscr{H}^p(Y,F)du$ via $du\wedge$ (cf. (1.2.4)), S^p_j also acts on $\mathscr{H}^p(Y,F)du$.

We recall that $C_j(\lambda) \in \mathrm{End}\big(\mathscr{H}^\bullet(Y,F[du])\big)$ $(j=1,2)$ is the scattering matrix associated with $\Omega^\bullet(Z_{j,\infty},F)$ (cf. §1.3.2). We recall that $C_j = C_j(0)$ and C^p_j is its restriction to $\mathscr{H}^p(Y,F) \oplus \mathscr{H}^{p-1}(Y,F)du$. By (1.2.45) and (1.2.46), we have

$$C_j = \begin{pmatrix} S^p_j & 0 \\ 0 & -S^{p-1}_j \end{pmatrix}\ . \tag{1.6.64}$$

Proposition 1.6.8. *The following identities hold*

$$\begin{aligned} T_{\mathscr{L}} &= \prod_{p=0}^{\dim Z} \det{}^*\Big(\frac{2 - S^p_1\circ S^p_2 - S^p_2\circ S^p_1}{4}\Big)^{\frac14(-1)^p} \\ &= \prod_{p=0}^{\dim Z} \det{}^*\Big(\frac{2 - C^p_{12} - (C^p_{12})^{-1}}{4}\Big)^{\frac14(-1)^p p}\ . \end{aligned} \tag{1.6.65}$$

Proof. The exact sequence (1.5.11) is the orthogonal sum of the following two exact sequences

(1.6.66)

$$\cdots \longrightarrow \mathscr{L}^p_{1,\mathrm{rel}} \cap \mathscr{L}^p_{2,\mathrm{rel}} \longrightarrow \mathscr{L}^p_1 \cap \mathscr{L}^p_2 \longrightarrow \mathscr{L}^p_{1,\mathrm{abs}} \cap \mathscr{L}^p_{2,\mathrm{abs}} \xrightarrow{\delta_{p,\mathscr{L}}} \cdots\ ,$$

$$\cdots \longrightarrow \mathscr{L}^p_{1,\mathrm{rel}} \cap (\mathscr{L}^p_{1,\mathrm{rel}} \cap \mathscr{L}^p_{2,\mathrm{rel}})^\perp \longrightarrow 0 \longrightarrow \mathscr{L}^p_{2,\mathrm{abs}} \cap (\mathscr{L}^p_{1,\mathrm{abs}} \cap \mathscr{L}^p_{2,\mathrm{abs}})^\perp \xrightarrow{\delta_{p,\mathscr{L}}} \cdots\ .$$

The $\delta_{p,\mathscr{L}}$ in the first line is zero. The other maps in the line are canonical injection/projection maps. By (1.6.60) and (1.6.62), the fist line in (1.6.66) does not contribute to $T_{\mathscr{L}}$. The second line in (1.6.66) splits into the short exact sequences

$$0 \longrightarrow \mathscr{L}^p_{2,\mathrm{abs}} \cap (\mathscr{L}^p_{1,\mathrm{abs}} \cap \mathscr{L}^p_{2,\mathrm{abs}})^\perp \xrightarrow{\delta_{p,\mathscr{L}}} \mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap (\mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap \mathscr{L}^{p+1}_{2,\mathrm{rel}})^\perp \longrightarrow 0\ . \tag{1.6.67}$$

By (1.2.46), the map $i_{\frac{\partial}{\partial u}} : \mathscr{H}^p(Y,F)du \to \mathscr{H}^p(Y,F)$ sends $\mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap (\mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap \mathscr{L}^{p+1}_{2,\mathrm{rel}})^\perp$ to $\mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap (\mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap \mathscr{L}^{p,\perp}_{2,\mathrm{abs}})^\perp$. We define the following commutative diagram with exact rows and isometric vertical maps

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{L}^p_{2,\mathrm{abs}} \cap (\mathscr{L}^p_{1,\mathrm{abs}} \cap \mathscr{L}^p_{2,\mathrm{abs}})^\perp \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap (\mathscr{L}^{p+1}_{1,\mathrm{rel}} \cap \mathscr{L}^{p+1}_{2,\mathrm{rel}})^\perp \arrow[r] \arrow[d, "i_{\frac{\partial}{\partial u}}"] & 0 \\ 0 \arrow[r] & \mathscr{L}^p_{2,\mathrm{abs}} \cap (\mathscr{L}^p_{1,\mathrm{abs}} \cap \mathscr{L}^p_{2,\mathrm{abs}})^\perp \arrow[r, dashed] & \mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap (\mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap \mathscr{L}^{p,\perp}_{2,\mathrm{abs}})^\perp \arrow[r] & 0\ . \end{tikzcd} \tag{1.6.68}$$

By (1.5.10), the map in the second row in (1.6.68) is the orthogonal projection. Since the vertical maps are isometric, the torsions of the first and second rows coincide.

Let P_p (resp. Q_p) be the orthogonal projection from $\mathscr{H}^p(Y,F)$ (resp. $\mathscr{H}^p(Y,F)$) onto $\mathscr{L}^p_{2,\mathrm{abs}}$ (resp. $\mathscr{L}^{p,\perp}_{1,\mathrm{abs}}$). Then

$$\begin{aligned}\mathscr{L}^p_{2,\mathrm{abs}} \cap (\mathscr{L}^p_{1,\mathrm{abs}} \cap \mathscr{L}^p_{2,\mathrm{abs}})^\perp &= \mathrm{im}(P_pQ_p)\ ,\\ \mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap (\mathscr{L}^{p,\perp}_{1,\mathrm{abs}} \cap \mathscr{L}^{p,\perp}_{2,\mathrm{abs}})^\perp &= \mathrm{im}(Q_pP_p)\ .\end{aligned} \tag{1.6.69}$$

We have the obvious identities

$$P_p = \frac{1}{2}(1+S^p_2)\ , \quad Q_p = \frac{1}{2}(1-S^p_1)\ . \tag{1.6.70}$$

By Lemma 1.6.7, (1.6.62) and (1.6.68)-(1.6.70), the torsion of (1.6.67) is given by

$$\det{}^*(1-P_p-Q_p+P_pQ_p+Q_pP_p)^{\frac{1}{4}} = \det{}^*\Big(\frac{2-S^p_1\circ S^p_2 - S^p_2\circ S^p_1}{4}\Big)^{\frac{1}{4}}\ . \tag{1.6.71}$$

By (1.6.59) and (1.6.60), $T_{\mathscr{L}}$ is the alternative product of the torsions of (1.6.67) for each p. Thus (1.6.71) implies the first equality in (1.6.65). We turn to prove the second one.

We denote

$$\begin{aligned} I_{p,\mathrm{abs}} &= \det{}^*\Big(\frac{2-S^p_1\circ S^p_2 - S^p_2\circ S^p_1}{4}\Big)^{\frac{1}{4}}\ ,\\ I_p &= \det{}^*\Big(\frac{2-C^p_{12}-(C^p_{12})^{-1}}{4}\Big)^{\frac{1}{4}}\ .\end{aligned} \tag{1.6.72}$$

It is sufficient to show that

$$\prod_p I^{(-1)^p}_{p,\mathrm{abs}} = \prod_p I_p^{(-1)^p p}\ . \tag{1.6.73}$$

By (1.6.64), we have

$$I_p = I_{p,\mathrm{abs}}\cdot I_{p+1,\mathrm{abs}}\ . \tag{1.6.74}$$

By (1.6.74), we have

$$\begin{aligned}\prod_p I^{(-1)^p}_{p,\mathrm{abs}} &= \prod_p I^{(-1)^p p}_{p,\mathrm{abs}} \prod_p I^{(-1)^{p-1}(p-1)}_{p,\mathrm{abs}}\\ &= \prod_p I^{(-1)^p p}_{p,\mathrm{abs}} \prod_p I^{(-1)^p p}_{p+1,\mathrm{abs}} = \prod_p I_p^{(-1)^p p}\ ,\end{aligned} \tag{1.6.75}$$

which gives exactly (1.6.73). The proof of Proposition 1.6.8 is completed. □

Proof of Theorem 1.0.2. We equip all the objects in (1.6.16) with metrics. All the metrics mentioned bellow are defined/recalled in §1.6.2.

- $\mathscr{H}^\bullet(Z_{12,\infty},F)$ is equipped with the metric $\|\cdot\|_{\mathscr{H}^\bullet(Z_{12,\infty},F),R}$;
- $\mathscr{H}^\bullet_{L^2}(Z_{12,\infty},F)\subseteq \mathscr{H}^\bullet(Z_{12,\infty},F)$ is equipped with the restricted metric ;
- $\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ $(j=1,2)$ is equipped with the metric $\|\cdot\|_{\mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F),R}$;
- $\mathscr{H}^\bullet_{L^2}(Z_{j,\infty},F)\subseteq \mathscr{H}^\bullet_{\mathbf{bd}}(Z_{j,\infty},F)$ is equipped with the restricted metric ;
- $\mathscr{L}^\bullet_1\cap\mathscr{L}^\bullet_2$ is equipped with the metric $\|\cdot\|_{\mathscr{L}^\bullet_1\cap\mathscr{L}^\bullet_2}$;
- $\mathscr{L}^\bullet_{j,\mathbf{bd}}$ $(j=1,2)$ is equipped with the metric $\|\cdot\|_{\mathscr{L}^\bullet_{j,\mathbf{bd}}}$.

Let $T_{\mathrm{h},j}$ $(j=1,2,3)$ be the torsion of the j-th row, $T_{\mathrm{v},j}$ $(j=1,\cdots,3n+3)$ be the torsion of the j-th column. By Proposition 1.6.2, we have

$$\mathscr{T}_R = \big(1+\mathscr{O}\big(e^{-cR}\big)\big)\,T_{\mathrm{h},2}\ . \tag{1.6.76}$$

By [BGS88a, Theorem 1.20], we have

$$T_{\mathrm{h},1}\, T_{\mathrm{h},2}^{-1}\, T_{\mathrm{h},3} = \prod_{k=1}^{3n+3} T_{\mathrm{v},k}^{(-1)^{k+1}} \,. \tag{1.6.77}$$

By Proposition 1.6.3, (1.6.59), (1.6.60) and (1.6.62), we have

$$\begin{aligned} T_{\mathrm{v},3p+1} &= \Big(1+\mathscr{O}\left(R^{-1}\right)\Big) R^{\frac{1}{2}\dim \mathscr{L}^p_{1,\mathbf{bd}}} \,, \\ T_{\mathrm{v},3p+2} &= \Big(1+\mathscr{O}\left(R^{-1}\right)\Big) (2R)^{\frac{1}{2}\dim \mathscr{L}^p_1\cap\mathscr{L}^p_2} \,, \\ T_{\mathrm{v},3p+3} &= \Big(1+\mathscr{O}\left(R^{-1}\right)\Big) R^{\frac{1}{2}\dim \mathscr{L}^p_{2,\mathbf{bd}}} \,. \end{aligned} \tag{1.6.78}$$

By (1.6.59), (1.6.60), (1.6.62) and the fact that the first row in (1.6.16) consists of canonical injection/projection maps, we have

$$T_{\mathrm{h},1} = 1 \,. \tag{1.6.79}$$

We recall that a_p, b_p and d_p are defined in (1.5.22). By Proposition 1.6.5, (1.6.59), (1.6.60) and (1.6.62), we have

$$T_{\mathrm{h},3} = \Big(1+\mathscr{O}\left(R^{-1}\right)\Big) \left(\prod_{p=1}^{n} 2^{(-1)^p a_p}\right) \left(\prod_{p=1}^{n} R^{(-1)^p d_p}\right) T_{\mathscr{L}} \,. \tag{1.6.80}$$

By the exactness of (1.5.11), we have

$$\sum_{p=1}^{n} (-1)^p \Big(\dim \mathscr{L}^p_{1,\mathbf{bd}} - \dim \mathscr{L}^p_1 \cap \mathscr{L}^p_2 + \dim \mathscr{L}^p_{2,\mathbf{bd}} \Big) = 0 \,, \tag{1.6.81}$$

$$\dim \mathscr{L}^p_1 \cap \mathscr{L}^p_2 = \dim \ker(\beta_{p,\mathscr{L}}) + \dim \mathrm{im}(\beta_{p,\mathscr{L}}) = a_p + b_p \,. \tag{1.6.82}$$

By (1.6.76) - (1.6.82), we get

$$\mathscr{T}_R = \Big(1+\mathscr{O}\left(R^{-1}\right)\Big) \left(\prod_{p=1}^{n} 2^{(-1)^p (a_p-b_p)/2}\right) \left(\prod_{p=1}^{n} R^{(-1)^p d_p}\right) T_{\mathscr{L}} \,. \tag{1.6.83}$$

By Lemma 1.5.2, Proposition 1.6.8 and (1.6.83), the proof of Theorem 1.0.2 is completed. □

1.7. **Gluing formula for the analytic torsion.**

In this section, we prove Theorem 1.0.3.

In §1.7.1, we review the Ray-Singer metric and the anomaly formula. In §1.7.2, applying Theorem 1.0.1, 1.0.2, we prove Theorem 1.0.3.

1.7.1. *Ray-Singer metric and Anomaly formula.*

Let X be a compact manifold (with or without boundary). Let (F, ∇^F) be a flat complex vector bundle over X.

We equip X with a Riemannian metric g^{TX}. We equip F with a Hermitian metric h^F. We suppose that g^{TZ} and h^F have a product structure near ∂X (cf. (1.0.1)).

We put absolute/relative boundary condition on ∂X. We recall that $H^\bullet_{\mathbf{bd}}(X,F)$ is defined by (1.1.4), and $\det H^\bullet_{\mathbf{bd}}(X,F)$ is the determinant of $H^\bullet_{\mathbf{bd}}(X,F)$, defined by (1.0.11).

We recall that $\Omega^\bullet_{\mathbf{bd}}(X,F)$ is defined by (1.1.5). Let $D^F_{X,\mathbf{bd}}$ be the Hodge-de Rham operator acting on $\Omega^\bullet_{\mathbf{bd}}(X,F)$, defined by (1.0.2). Let $\|\cdot\|_{\det H^\bullet_{\mathbf{bd}}(X,F)}$ be the L^2-metric on $\det H^{\bullet}_{\mathbf{bd}}(X,F)$ induced by Hodge Theorem (cf. Theorem 1.1.1). Let $\zeta(s)$ be the ζ-function of $D^{F,2}_X$, defined by (1.0.6).

Definition 1.7.1. The Ray-Singer metric on $\det H^{\bullet}_{\mathbf{bd}}(X,F)$ is defined as follows,

$$\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}_{\mathbf{bd}}(X,F)} = \|\cdot\|_{\det H^{\bullet}_{\mathbf{bd}}(X,F)} \exp\left(\frac{1}{2}\zeta'(0)\right) . \tag{1.7.1}$$

Let $g^{TX\prime}$ be another Riemannian metric on X. We suppose that g^{TX} and $g^{TX\prime}$ coincide on a neighborhood of ∂X. Let $\|\cdot\|^{\mathrm{RS}\prime}_{\det H^{\bullet}_{\mathbf{bd}}(X,F)}$ be the Ray-Singer metric associated with $g^{TX\prime}$ and h^F. Before stating the anomaly formula calculating the ratio of $\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}_{\mathbf{bd}}(X,F)}$ and $\|\cdot\|^{\mathrm{RS}\prime}_{\det H^{\bullet}_{\mathbf{bd}}(X,F)}$, we define the Euler form and its Chern-Simons form.

Let $o(TX)$ be the orientation bundle of TX. Let ∇^{TX} be the Levi-Civita connection on TX. Let $R^{TX} = \left(\nabla^{TX}\right)^2$ be its curvature. We define its Euler form (cf. [BZ92, (4.9)])

$$e(TX,\nabla^{TX}) = \mathrm{Pf}\left[\frac{R^{TX}}{2\pi}\right] \in \Omega^{\dim X}\big(X, o(TX)\big) . \tag{1.7.2}$$

Let $\left(g^{TX}_s\right)_{s\in[0,1]}$ be a smooth family of Riemannian metrics on TX such that $g^{TX}_0 = g^{TX}$, $g^{TX}_1 = g^{TX\prime}$. Moreover, we suppose that all the g^{TX}_s coincide on a neighborhood of ∂X. Let ∇^{TX}_s be the Levi-Civita connection associated with g^{TX}_s. Set

$$\begin{aligned}&\widetilde{e}\left(TX, \left(\nabla^{TX}_s\right)_{s\in[0,1]}\right)\\ &= \int_0^1 \left\{\frac{\partial}{\partial b}\Big|_{b=0} \mathrm{Pf}\left[\frac{1}{2\pi}\left(\nabla^{TX}_s\right)^2 + \frac{b}{2\pi}\left(\frac{\partial}{\partial s}\nabla^{TX}_s - \frac{1}{2}\left[\nabla^{TX}_s, \left(g^{TX}_s\right)^{-1}\frac{\partial}{\partial s}g^{TX}_s\right]\right)\right]\right\} ds .\end{aligned} \tag{1.7.3}$$

By [BZ92, (4.10)], we have

$$d\,\widetilde{e}\left(TX, \left(\nabla^{TX}_s\right)_{s\in[0,1]}\right) = e(TX,\nabla^{TX\prime}) - e(TX,\nabla^{TX}) . \tag{1.7.4}$$

We are in a special case of [BM06, Theorem 1.9] : since g^{TX}_s coincide near ∂X, the boundary term $\widetilde{e}_b$ in [BM06, (1.45)] vanishes, then the image of $\widetilde{e}\left(TX, \left(\nabla^{TX}_{s'}\right)_{s'\in[0,1]}\right)$ in

$$\Omega^{\dim X-1}\big(X, o(TX)\big)\Big/\Big\{d\alpha \ : \ \alpha\in\Omega^{\dim X-2}\big(X,o(TX)\big) \ , \ \mathrm{supp}(\alpha)\cap\partial X = \emptyset\Big\} , \tag{1.7.5}$$

denoted by $\widetilde{e}\left(TX,\nabla^{TX},\nabla^{TX\prime}\right)$, is independent of the path $\left(\nabla^{TX}_s\right)_{s\in[0,1]}$, which may be identified with the secondary Euler class in [BM06, Theorem 1.9].

We define

$$\theta(F,h^F) = \mathrm{Tr}\left[\left(h^F\right)^{-1}\nabla^F h^F\right] \in \Omega^1(X) , \tag{1.7.6}$$

which is closed (cf. [BZ92, Proposition 4.6]).

The following theorem is a consequence of the anomaly formula for manifolds with boundary [BM06, Theorem 0.1], which extends the anomaly formula for closed manifolds [BZ92, Theorem 0.1].

Theorem 1.7.2. *We have*

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}\prime}_{\det H^{\bullet}_{\mathbf{bd}}(X,F)}}{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}_{\mathbf{bd}}(X,F)}}\right)^2 = -\int_X \theta(F,h^F)\,\widetilde{e}(TX,\nabla^{TX},\nabla^{TX\prime}) . \tag{1.7.7}$$

1.7.2. *Gluing formula : proof of Theorem 1.0.3.*

We use the notations and assumptions in §1.3.1. We recall that $\varrho \in \lambda(F)$ is defined by (1.0.18). In the same way, we define

$$\varrho_R \in \lambda_R(F) := \Big(\det H^\bullet(Z_R,F)\Big)^{-1} \otimes \det H^\bullet_{\mathbf{bd}}(Z_{1,R},F) \otimes \det H^\bullet_{\mathbf{bd}}(Z_{2,R},F) \ . \tag{1.7.8}$$

The commutative diagram (1.6.2) induces an isomorphism $\varphi_{R*} : \lambda_R(F) \to \lambda(F)$. By the functoriality of the construction of ϱ, we have

$$\varphi_{R*}\varrho_R = \varrho \ . \tag{1.7.9}$$

Let $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z_R,F)}$ be the Ray-Singer metric on $\det H^\bullet(Z_R,F)$. Let $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_{j,R},F)}$ $(j=1,2)$ be the Ray-Singer metric on $\det H^\bullet_{\mathbf{bd}}(Z_{j,R},F)$. Let $\|\cdot\|^{\mathrm{RS}}_{\lambda_R(F)}$ be the induced metric on $\lambda_R(F)$.

Lemma 1.7.3. *For $R>0$, we have*

$$\|\varrho_R\|^{\mathrm{RS}}_{\lambda_R(F)} = \|\varrho\|^{\mathrm{RS}}_{\lambda(F)} \ . \tag{1.7.10}$$

Proof. We use the convention $Z_0 = Z$ and $Z_{0,R} = Z_R$. We identify $H^\bullet_{\mathbf{bd}}(Z_{j,R},F)$ $(j = 0,1,2)$ to $H^\bullet_{\mathbf{bd}}(Z_j,F)$ via φ_{R*}. By (1.7.8) and (1.7.9), it is equivalent to show that, for $R' > R \geqslant 0$,

$$\sum_{j=0}^{2}(-1)^{(j-1)(j-2)/2}\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F),R'}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F),R}}\right)^2 = 0 \ . \tag{1.7.11}$$

Let $\nabla^{TZ_{j,R}}$ $(j=1,2)$ be the Levi-Civita connections on $TZ_{j,R}$. We recall that the diffeormorphism $\tilde{\varphi}_{R,R'} : Z_R \to Z_{R'}$ is constructed in the proof of Proposition 1.3.3. By restricting to $Z_{j,R}$, $\tilde{\varphi}_{R,R'}$ induces an diffeormphism $\tilde{\varphi}_{R,R'} : Z_{j,R} \to Z_{j,R'}$ $(j=1,2)$. We choose $g^{TZ_R}_s = (1-s)g^{TZ_R} + s\tilde{\varphi}^*_{R,R'}g^{TZ_{R'}}$. Let $g^{TZ_{j,R}}_s$ $(j=1,2)$ be the restricted metric on $Z_{j,R}$. Let $\nabla^{TZ_{j,R}}_s$ $(j=0,1,2)$ be the associated Levi-Civita connections. By (1.7.3), for $j=1,2$,

$$\widetilde{e}\Big(TZ_R,\big(\nabla^{TZ_R}_s\big)_{s\in[0,1]}\Big)\Big|_{Z_{j,R}} = \widetilde{e}\Big(TZ_{j,R},\big(\nabla^{TZ_{j,R}}_s\big)_{s\in[0,1]}\Big) \ . \tag{1.7.12}$$

Since $\tilde{\varphi}_{R,R'}$ preserves the metric near the boundary, by (1.7.7), we get, for $j=0,1,2$,

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F),R'}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F),R}}\right)^2 = -\int_{Z_{j,R}}\theta(F,h^F)\,\widetilde{e}\Big(TZ_{j,R},\big(\nabla^{TZ_{j,R}}_s\big)_{s\in[0,1]}\Big) \ . \tag{1.7.13}$$

By (1.7.12) and (1.7.13), we get (1.7.11). □

Proof of Theorem 1.0.3. Recall that $\zeta_{1,R}(s)$, $\zeta_{2,R}(s)$ and $\zeta_R(s)$ are defined in §1.0.2, and $\mathscr{T}_R$ is defined in §1.0.3. By (1.7.1), it is sufficient to show that

$$\mathscr{T}_R\exp\left(\frac{1}{2}\zeta'_{1,R}(0)+\frac{1}{2}\zeta'_{2,R}(0)-\frac{1}{2}\zeta'_R(0)\right) = 2^{-\frac{1}{2}\chi(Y,F)} \ . \tag{1.7.14}$$

By Theorem 1.0.1, 1.0.2, the left hand side of (1.7.14) tends to $2^{-\frac{1}{2}\chi(Y,F)}$ as $R\to\infty$. Meanwhile, by Lemma 1.7.3, the left hand side of (1.7.14) is independent of R. This proves (1.7.14). □

1.8. **Appendix : Matrix valued holomorphic functions.**

Let $(V, \langle\cdot,\cdot\rangle)$ be a Hermitian vector space of dimension m. Let $\|\cdot\|$ be the norm induced by $\langle\cdot,\cdot\rangle$. Let $D \subseteq \mathbb{C}$ be an open disc centered at 0. Let $C : D \to \mathrm{End}(V)$ be a holomorphic function such that, for any $z \in D \cap \mathbb{R}$, $C(z)$ is a unitary matrix.

The following theorem is proved in [K95, §2.6, Theorem 6.1].

Theorem 1.8.1. *There exist real holomorphic functions, i.e., their expansions at* 0 *are of real coefficients,* $\theta_1(z), \cdots, \theta_m(z)$ *in the neighborhood of* 0 *such that* $e^{i\theta_1(z)}, \cdots, e^{i\theta_m(z)}$ *give all the eigenvalues of* $C(z)$.

Furthermore, there exist $P_1(z), \cdots, P_m(z) \in \mathrm{End}(V)$, *which are defined for* z *in the neighborhood of* 0 *and holomorphic on* z *such that* $P_j(z)$ *is the orthogonal projection to the eigenspace associated with* $\theta_j(z)$, *i.e.,*

$$\begin{aligned} 1 &= P_1(z) + \cdots + P_m(z) \ , \\ P_j(z)P_k(z) &= 0 \ , \quad \text{for } 1 \leqslant j,k \leqslant m \ , \ j \neq k \ , \\ C(z) &= e^{i\theta_1(z)}P_1(z) + \cdots + e^{i\theta_m(z)}P_m(z) \ . \end{aligned} \tag{1.8.1}$$

In the sequel, by shrinking D to a smaller disc if necessary, we suppose that θ_j and P_j $(j = 1, \cdots, m)$ are all-well defined in the neighborhood of $\overline{D}$.

For $R > 0$, we consider the equation

$$e^{4iRz}C(z)v = v \ , \tag{1.8.2}$$

where $z \in D$, $v \in V$. By Theorem 1.8.1, for R and z fixed, (1.8.2) as an equation of v has non trivial solution if and only if one of $4Rz + \theta_1(z), \cdots, 4Rz + \theta_m(z)$ lies in $2\pi\mathbb{Z}$.

Proposition 1.8.2. *There exist* $R_0 > 0$, $\varepsilon > 0$ *such that for* $R > R_0$, $z_0 \in]-\varepsilon, \varepsilon[$, $v \in V$, *if*

$$\left\| e^{4iRz_0}C(z_0)v - v \right\| < \|v\| \ , \tag{1.8.3}$$

then there exist $z_1, \cdots, z_m \in \mathbb{R}$, $w_1, \cdots, w_m \in V$ *satisfying*

$$\begin{aligned} |z_j - z_0|^2 &< \|v\|^{-1} \cdot \left\| e^{4iRz}C(z_0)v - v \right\| \ , \\ \|P_j(z_0)v - w_j\|^2 &< \|v\| \cdot \left\| e^{4iRz}C(z_0)v - v \right\| \ , \\ e^{4iRz_j}C(z_j)w_j - w_j &= 0 \ , \end{aligned} \tag{1.8.4}$$

for $j = 1, \cdots, m$.

Proof. We equip $\mathrm{End}(V)$ with the operator norm.

We fix $B_1, B_2 > 0$ such that for any $s, t \in \overline{D}$ and $j = 1, \cdots, m$,

$$|\theta_j(s) - \theta_j(t)| < B_1 |s - t| \ , \quad \|P_j(s) - P_j(t)\| < B_2 |s - t| \ . \tag{1.8.5}$$

We choose $\varepsilon > 0$, $R_0 > 0$ such that

$$\begin{aligned} &\left] -\varepsilon - \frac{2\pi}{4R_0 - B_1}, \varepsilon + \frac{2\pi}{4R_0 - B_1} \right[\subseteq D \ , \\ &0 < \frac{2}{4R_0 - B_1} < 1 \ , \quad 0 < \frac{2B_2}{4R_0 - B_1} < 1 \ . \end{aligned} \tag{1.8.6}$$

Set $v_j = P_j(z_0)v$. By (1.8.1), for $R > R_0$, we have

$$e^{4iRz_0}C(z_0)v - v = \sum_{j=1}^m \left(e^{4iRz_0 + i\theta_j(z_0)} - 1 \right) v_j \ , \tag{1.8.7}$$

Since these v_j are mutually orthogonal, we have

$$\left|e^{4iRz_0+i\theta_j(z_0)}-1\right|\cdot\|v_j\|\leqslant\left\|e^{4iRz_0}C(z_0)v-v\right\| . \tag{1.8.8}$$

If $\|v_j\|^2<\|v\|\cdot\left\|e^{4iRz}C(z_0)v-v\right\|$, set $w_j=0$, $z_j=z_0$. Then (1.8.4) holds trivially. Otherwise, by (1.8.3) and (1.8.8), we have

$$\left|e^{4iRz_0+i\theta_j(z_0)}-1\right|^2\leqslant\|v\|^{-1}\cdot\left\|e^{4iRz_0}C(z_0)v-v\right\|<1 . \tag{1.8.9}$$

Then there exists $k_j\in\mathbb{Z}$ such that

$$\left|4Rz_0+\theta_j(z_0)-2k_j\pi\right|^2\leqslant 4\,\|v\|^{-1}\cdot\left\|e^{4iRz_0}C(z_0)v-v\right\| . \tag{1.8.10}$$

For $R>R_0$, by (1.8.5) and (1.8.6), $4Rz+\theta_j(z)-2k_j\pi$ as a function of $z\in\mathbb{R}$ is strictly increasing. Moreover, its derivative is greater than $4R-B_1$. Let $z_j\in\mathbb{R}$ be the unique real number satisfying $4Rz_j+\theta_j(z_j)-2k_j\pi=0$, then

$$|z_j-z|^2<\left(\frac{2}{4R-B_1}\right)^2\|v\|^{-1}\cdot\left\|e^{4iRz_0}C(z_0)v-v\right\| . \tag{1.8.11}$$

By (1.8.6) and (1.8.11), the first equation in (1.8.4) holds. Set $w_j=P(z_j)v$, then the third equation in (1.8.4) holds trivially. Furthermore, by the choice of B_2, we have

$$\begin{aligned}\|P_j(z_0)v-w_j\|&=\left\|\big(P_j(z_0)-P_j(z_j)\big)v\right\|\\&\leqslant\|P_j(z_0)-P_j(z_j)\|\cdot\|v\|\leqslant B_2\,|z_0-z_j|\cdot\|v\| .\end{aligned} \tag{1.8.12}$$

By (1.8.6), (1.8.11) and (1.8.12), the second equation in (1.8.4) holds. □

For $R>0$, set

$$\begin{aligned}\Lambda_R(C)&=\left\{\rho>0\ :\ \det\left(e^{4iR\rho}C(\rho)-1\right)=0\right\} ,\\\Lambda_R^*(C)&=\left\{\lambda>0\ :\ \det\left(e^{4iR\lambda}C(0)-1\right)=0\right\} .\end{aligned} \tag{1.8.13}$$

We fix $\kappa>0$.

Proposition 1.8.3. *There exist $a>0$, $R_0>0$ such that for any $R>R_0$, $R^{-1+\kappa}\leqslant\gamma\leqslant 1$ and $f\in\mathscr{C}^1(\mathbb{R})$, we have*

$$\left|\sum_{\rho\in\Lambda_R(C)\,,\,|\rho|<\gamma}f(\rho)-\sum_{\lambda\in\Lambda_R^*(C)\,,\,|\lambda|<\gamma}f(\lambda)\right|\leqslant a\gamma^2\sup_{|x|\leqslant\gamma}|f'(x)|+a\gamma\sup_{|x|\leqslant\gamma}|f(x)| . \tag{1.8.14}$$

Proof. By Theorem 1.8.1, we may suppose that $C(\rho)=e^{i\theta(\rho)}$, where θ is an analytic function. The rest of the proof is a direct estimate, and we leave it to readers. □

Set

$$\zeta_{C,R}(s)=-\sum_{\lambda\in\Lambda_R^*(C)}\left(\lambda^2\right)^{-s} . \tag{1.8.15}$$

We recall that $m=\dim V$. Set $r=\dim\ker\big(C(0)-1\big)$.

Proposition 1.8.4. *If $\mathrm{Sp}\big(C(0)\big)=\overline{\mathrm{Sp}\big(C(0)\big)}$, then*

$${\zeta_{C,R}}'(0)=r\log(2R)+m\log 2+\frac{1}{2}\log\det{}^*\left(\frac{2-C(0)-C(0)^{-1}}{4}\right) . \tag{1.8.16}$$

Proof. As special cases of the Hurwitz ζ-functions (cf. [W99, §7]), we have

$$-\frac{\partial}{\partial s}\Big|_{s=0} \sum_{k=1}^{\infty} \left(\frac{2\pi k - \theta}{4R}\right)^{-2s} = \begin{cases} \log(4R) & \text{for } \theta = 0\ , \\ \frac{1}{2}\log(2 - 2\cos\theta) & \text{for } 0 < \theta \leqslant \pi\ . \end{cases} \tag{1.8.17}$$

Since $C(0)$ is diagonalizable, it suffices to consider the consider the following cases.

Case 1. $m = 1$, $r = 1$, $C = 1$, then (1.8.16) is equivalent to (1.8.17) with $\theta = 0$.

Case 2. $m = 1$, $r = 0$, $C = -1$, then (1.8.16) is equivalent to (1.8.17) with $\theta = \pi$.

Case 3. $m = 2$, $r = 0$, $\mathrm{Sp}C = \left\{e^{i\alpha}, e^{-i\alpha}\right\}$ with $\alpha \in]0, \pi[$, then (1.8.16) is equivalent to (1.8.17) with $\theta = \alpha$. $\square$

2. Riemann-Roch-Grothendieck and flat complex fibrations

2.0. Introduction.

The real and complex analytic torsions were introduced by Ray-Singer [RS71, RS73]. For a compact real (resp. complex) manifold equipped with a Riemannian (resp. Hermitian) metric and a flat (resp. holomorphic) Hermitian vector bundle, its real (resp. complex) analytic torsion is a spectral invariant of the Laplacian.

Cheeger [Che79] and Müller [M78] proved independently that the real analytic torsion is a topological invariant for unitarily flat vector bundles. Müller [M93] also extended their result to unimodular flat vector bundles. In the general case, the dependence of the real analytic torsion on the metrics was calculated by Bismut-Zhang [BZ92], who also established an extension of the Cheeger-Müller theorem in the general case.

For a real smooth fibration $\pi : M \to S$ with compact fiber X, and a flat complex vector bundle F over M, Bismut and Lott [BL95] gave a R.R.G. formula for the odd Chern classes of the direct image $R^{\cdot}\pi_* F$, which is a flat vector bundle over S, in terms of the Euler class of the relative tangent bundle TX and the corresponding odd Chern classes of F. When equipping the considered vector bundles with metrics, these classes can be represented by explicit differential forms. By transgressing the equality of cohomology classes at the level of differential forms, they also obtained even analytic torsion forms on S, whose coboundary is equal to the difference between the differential forms appearing on the left and right hand side of the R.R.G. formula. The parallel work for holomorphic fibrations extending the complex analytic torsion was done by Bismut-Gillet-Soulé [BGS88b] and Bismut-Köhler [BK92].

In this article, we consider a flat fibration $q : \mathcal{N} \to M$ with complex fiber N and a complex vector bundle E over $\mathcal{N}$ which is holomorphic along N and flat along horizontal directions in $\mathcal{N}$. First, we give a R.R.G. formula for the odd Chern classes of $R^{\cdot}\pi_* F$ in terms of the Todd class of the relative tangent bundle and of the Chern classes of F. By equipping the various vector bundles with Hermitian metrics, we construct even analytic torsion forms on M which transgress the equality of the corresponding cohomology classes.

In a second part, we combine the techniques of Bismut-Lott [BL95] and of the first part. We consider the projection $r : \mathcal{N} \to S$ with fiber Y, and the corresponding family of bicomplexes equipped with the chain map $d_X + \overline{\partial}_N$. When introducing suitable Hermitian metrics, we construct analytic torsion forms on S associated with this bicomplex.

We also consider the case where L is a line bundle, equipped with a Hermitian metric g^L such that the curvature of the corresponding fiberwise Chern connection r^L is positive along the fibers. We introduce a suitable nondegeneracy assumption on the metric g^L from Bismut-Ma-Zhang [BMaZ11, BMaZ15] that guarantees that for $p \in \mathbb{N}$ large enough, the de Rham cohomology of $q_*(E \otimes L^p)$ along the fibers X vanishes identically. In this case, we construct even analytic torsion forms on the S that are associated with the above bicomplex.

In a last step, we give a formula relating the analytic torsion forms of the above bicomplex to the analytic torsion forms of Bismut-Lott for $q_*(E \otimes L^p)$ and the analytic torsion forms of the first part of the article.

Let us now give more detail on the content of the present article.

2.0.1. *Chern-Weil theory and its extensions.*

Let M be a smooth manifold. Given a complex vector bundle E of rank r over M, a connection ∇^E on E and an invariant polynomial P on $\mathfrak{gl}(r, \mathbb{C})$, Chern-Weil theory assigns a closed differential form of even degree

$$P(E,\nabla^E) \in \Omega^{\mathrm{even}}(M)\ , \tag{2.0.1}$$

whose cohomology class $\big[P(E,\nabla^E)\big] \in H^{\mathrm{even}}(M)$ does not depend on ∇^E, and will be denoted by $P(E)$. This theory will be referred to as the even Chern-Weil theory.

If ∇^E is a flat connection, i.e., $\nabla^{E,2}=0$, $P(E,\nabla^E)$ is a constant function.

A Chern-Weil theory for flat vector bundles was developed by Bismut-Lott [BL95, §1]. Given a flat complex vector bundle (E,∇^E) over M, a Hermitian metric g^E on E and an odd polynomial f, we assign a closed differential form of odd degree

$$f(E,\nabla^E,g^E) \in \Omega^{\mathrm{odd}}(M)\ , \tag{2.0.2}$$

whose cohomology class $\big[f(E,\nabla^E,g^E)\big] \in H^{\mathrm{odd}}(M)$ is independent of g^E, and will be denoted by $f(E,\nabla^E)$. This theory will be referred to as the odd Chern-Weil theory.

In this article , we will construct characteristic classes for flat fibrations with complex fibers. Our construction is a mixture of the even and odd Chern-Weil theory.

Let G be a Lie group. Let $p : P_G \to M$ be a flat G-principal bundle. Let N be a compact complex manifold. We assume that G acts holomorphically on N. Set

$$\mathcal{N} = P_G \times_G N\ . \tag{2.0.3}$$

Let

$$q : \mathcal{N} \to M \tag{2.0.4}$$

be the canonical projection. Then q induces a flat fibration with canonical fiber N.

Let E_0 be a holomorphic vector bundle over N. We assume that the action of G lifts holomorphically to E_0. Set

$$E = P_G \times_G E_0\ . \tag{2.0.5}$$

Then E is a complex vector bundle over $\mathcal{N}$.

In §2.2, for such a vector bundle E and a Hermitian metric g^E on E, we assign odd differential forms as follows. Set

$$\begin{aligned} \Omega^{\cdot}(M) &= \mathscr{C}^\infty\big(M,\Lambda^{\cdot}(T^*M)\big)\ ,\\ \Omega^{\cdot}(\mathcal{N},E) &= \mathscr{C}^\infty\big(\mathcal{N},\Lambda^{\cdot}(T^*\mathcal{N})\otimes E\big)\ . \end{aligned} \tag{2.0.6}$$

Let d_M be the de Rham operator on $\Omega^{\cdot}(M)$. Let d^E_M be the lift of d_M to $\Omega^{\cdot}(\mathcal{N},E)$. Set

$$\omega^E = \big(g^E\big)^{-1} d^E_M\, g^E \in \Omega^1(\mathcal{N},\mathrm{End}(E))\ . \tag{2.0.7}$$

Let ∇^E_N be the fiberwise Chern connection on (E,g^E). Let A^E be the unitary connection on E defined by

$$A^E = \nabla^E_N + d^E_M + \frac{1}{2}\omega^E\ . \tag{2.0.8}$$

Let r be the rank of r. Let $\mathfrak{gl}(r,\mathbb{C})$ be the Lie algebra of $GL(r,\mathbb{C})$. Let $N^{\Lambda^{\cdot}(T^*\mathcal{N})}$ be the number operator on $\Lambda^{\cdot}(T^*\mathcal{N})$, i.e., for $\alpha \in \Lambda^k(T^*\mathcal{N})$, $N^{\Lambda^{\cdot}(T^*\mathcal{N})}\alpha = k\alpha$. For an invariant polynomial P on $\mathfrak{gl}(r,\mathbb{C})$ (under the conjugate action of $GL(r,\mathbb{C})$), put

$$\begin{aligned} P(E,g^E) &= (2\pi i)^{-\frac{1}{2}N^{\Lambda^{\cdot}(T^*\mathcal{N})}} P(-A^{E,2}) \in \Omega^{\mathrm{even}}(\mathcal{N})\ ,\\ \widetilde{P}(E,g^E) &= (2\pi i)^{\frac{1}{2}-\frac{1}{2}N^{\Lambda^{\cdot}(T^*\mathcal{N})}} \Big\langle P'(-A^{E,2}), \frac{\omega^E}{2}\Big\rangle \in \Omega^{\mathrm{odd}}(\mathcal{N})\ . \end{aligned} \tag{2.0.9}$$

Theorem 2.0.1. *The differential form*

$$q_*\big[P(E,g^E)\big] \in \Omega^{\mathrm{even}}(M) \tag{2.0.10}$$

is a constant function.

The differential form

$$q_*\big[\widetilde{P}(E,g^E)\big] \in \Omega^{\mathrm{odd}}(M) \tag{2.0.11}$$

is closed. Its cohomology class

$$\Big[q_*\big[\widetilde{P}(E,g^E)\big]\Big] \in H^{\mathrm{odd}}(M) \tag{2.0.12}$$

is independent of g^E.

In the sequel, we use the notation

$$q_*\big[\widetilde{P}(E)\big] = \Big[q_*\big[\widetilde{P}(E,g^E)\big]\Big] \in H^{\mathrm{odd}}(M)\ . \tag{2.0.13}$$

Let F be another vector bundle (of rank r') over $\mathcal{N}$ satisfying the same properties as E. Let g^F be a Hermitian metric on F. Let Q be an invariant polynomial on $\mathfrak{gl}(r',\mathbb{C})$. The natural product on the forms $\widetilde{P}(E,g^E)$ and $\widetilde{Q}(F,g^F)$ is given by

$$\widetilde{P}(E,g^E) * \widetilde{Q}(F,g^F) = \widetilde{P}(E,g^E)Q(F,g^F) + P(E,g^E)\widetilde{Q}(F,g^F)\ . \tag{2.0.14}$$

2.0.2. *A R.R.G. theorem for flat fibrations with complex fibers.*

In the sequel, we suppose that N is a Kähler manifold.

Let $H^{\cdot}(N,E)$ be the fiberwise Dolbeault cohomology group of E along N. Then $H^{\cdot}(N,E)$ is a graded flat vector bundle over M. Let $\nabla^{H^{\cdot}(N,E)}$ be its flat connection.

Let $f(x) = x\exp(x^2)$.

Let

$$f\big(H^{\cdot}(N,E),\nabla^{H^{\cdot}(N,E)}\big) \in H^{\mathrm{odd}}(M,\mathbb{R}) \tag{2.0.15}$$

be the Bismut-Lott odd characteristic class [BL95, §1].

We establish the following Riemann-Roch-Grothendieck formula.

Theorem 2.0.2. *We have*

$$f\big(H^{\cdot}(N,E),\nabla^{H^{\cdot}(N,E)}\big) = q_*\big[\widetilde{\mathrm{Td}}(TN) * \widetilde{\mathrm{ch}}(E)\big] \in H^{\mathrm{odd}}(M,\mathbb{R})\ . \tag{2.0.16}$$

Here $\widetilde{\mathrm{Td}}(TN) * \widetilde{\mathrm{ch}}(E)$ is defined by (2.0.9) and (2.0.14).

Now we explain the idea of the proof. We use the superconnection formalism [BL95, §2].

Put

$$\mathscr{E} = \mathscr{C}^\infty(N,\Lambda^{\cdot}(\overline{T^*N})\otimes E)\ . \tag{2.0.17}$$

Then $\mathscr{E}$ is an infinite dimensional flat vector bundle over M. Let $d_M^{\mathscr{E}}$ be its flat connection. Let $\overline{\partial}_N^E$ be the Dolbeault operator acting on $\mathscr{E}$. Set

$$A^{\mathscr{E}} = \overline{\partial}_N^E + d_M^{\mathscr{E}}\ . \tag{2.0.18}$$

Then $A^{\mathscr{E}}$ acts on $\Omega^{\cdot}(M,\mathscr{E})$. Also $A^{\mathscr{E}}$ is a flat superconnection on $\mathscr{E}$ in the sense of Bismut-Lott [BL95, Definition 1.1].

Let g^{TN} be a fiberwise Kähler metric on TN. Let g^E be a Hermitian metric on E. Let $g^{\mathscr{E}}$ be the induced L^2-metric on $\mathscr{E}$. Let $A^{\mathscr{E},*}$ be the adjoint superconnection of $A^{\mathscr{E}}$ in the sense of Bismut-Lott [BL95, Definition 1.6].

Let $N^{\Lambda^\cdot(T^*M)}$ be the number operator of $\Lambda^\cdot(T^*M)$. Set

$$D^{\mathscr{E}} = 2^{-N^{\Lambda^\cdot(T^*M)}}\big(A^{\mathscr{E},*} - A^{\mathscr{E}}\big)2^{N^{\Lambda^\cdot(T^*M)}} \in \Omega^\cdot(M, \mathrm{End}(\mathscr{E})) \; . \tag{2.0.19}$$

For $t>0$, let $D^{\mathscr{E}}_t$ be the $D^{\mathscr{E}}$ associated with the metrics $\frac{1}{t}g^{TN}$ and g^E. Following Bismut-Lott [BL95, (2.22),(2,23)], we define

$$\begin{aligned}\alpha_t &= (2\pi i)^{\frac{1}{2}-\frac{1}{2}N^{\Lambda^\cdot(T^*M)}}\,\mathrm{Tr_s}\Big[D^{\mathscr{E}}_t \exp(D^{\mathscr{E},2}_t)\Big] \; ,\\ \beta_t &= (2\pi i)^{-\frac{1}{2}N^{\Lambda^\cdot(T^*M)}}\,\mathrm{Tr_s}\Big[\frac{N^{\Lambda^\cdot(\overline{T^*N})}}{2}(1+2D^{\mathscr{E},2}_t)\exp(D^{\mathscr{E},2}_t)\Big] \; ,\end{aligned} \tag{2.0.20}$$

and we show that

$$d_M\alpha_t = 0 \; , \qquad \frac{\partial}{\partial t}\alpha_t = \frac{1}{t}d_M\beta_t \; . \tag{2.0.21}$$

Let $g^{H^\cdot(N,E)}$ be the metric on $H^\cdot(N,E)$ induced by the L^2-metric on $\mathscr{E}$ via the Hodge theorem. Let

$$f\big(H^\cdot(N,E), \nabla^{H^\cdot(N,E)}, g^{H^\cdot(N,E)}\big) \in \Omega^{\mathrm{odd}}(M) \tag{2.0.22}$$

be the Bismut-Lott odd characteristic form [BL95, Definition 1.7].

Theorem 2.0.2 is a consequence of the following theorem.

Theorem 2.0.3. *We have*

(2.0.23)

$$\begin{aligned}&\alpha_t = f\big(H^\cdot(N,E), \nabla^{H^\cdot(N,E)}, g^{H^\cdot(N,E)}\big) + \mathscr{O}\big(\frac{1}{\sqrt{t}}\big) \; , \qquad \textit{as } t\to\infty \; ,\\ &\alpha_t = q_*\big[\widetilde{\mathrm{Td}}(TN,g^{TN}) * \widetilde{\mathrm{ch}}(E,g^E)\big] + \frac{\textit{a fixed exact form}}{t} + \mathscr{O}\big(\sqrt{t}\big) \; , \qquad \textit{as } t\to 0 \; .\end{aligned}$$

2.0.3. *An analytic torsion form for flat fibration with complex fibers.*

As a consequence of Theorem 2.0.3, we obtain an analytic torsion form, which generalizes the Ray-Singer analytic torsion for complex manifolds [RS73].

In the same way as in (2.0.23), we also obtain an asymptotic estimate for β_t as $t\to\infty$ and $t\to 0$. With the help of this estimate, we construct explicitly a differential form

$$\mathscr{T}(g^{TN}, g^E) \in \Omega^{\mathrm{even}}(M) \; , \tag{2.0.24}$$

which is defined by subtracting the singularities of the following integral

$$-\int_0^\infty \beta_t \frac{dt}{t} \; . \tag{2.0.25}$$

Moreover, by the asymptotic estimate for β_t, the singularities of the integral consist of closed forms. Now, applying (2.0.21) and (2.0.23), we get

$$\begin{aligned}&d_M\mathscr{T}(g^{TN}, g^E)\\ &= q_*\big[\widetilde{\mathrm{Td}}(TN,g^{TN}) * \widetilde{\mathrm{ch}}(E,g^E)\big] - f\big(H^\cdot(N,E), \nabla^{H^\cdot(N,E)}, g^{H^\cdot(N,E)}\big) \; .\end{aligned} \tag{2.0.26}$$

Moreover, we show that the degree zero component of $\mathscr{T}(g^{TN}, g^E)$ is the Ray-Singer holomorphic torsion associated with (N, g^{TN}, E, g^E).

2.0.4. *The analytic torsion forms of Bismut-Lott and their extension.*

The Bismut-Lott analytic torsion forms [BL95, Definition 3.22] extends the Ray-Singer analytic torsion for real manifolds [RS71]. We briefly summarize the results of [BL95].

Let $\pi : M \to S$ be a real smooth fibration with compact fiber X. Let $T^H M \subseteq TM$ be a lift of TS, i.e., the restriction of the map $TM \to \pi^* TS$ to $T^H M$ is an isomorphism. Let TX be a Riemannian metric on TX. Let ∇^{TX} be the Levi-Civita connection on TX (cf. §2.1.4).

Let (F, ∇^F) be a flat complex vector bundle over M. Let g^F be a Hermitian metric on F. Let $H^\cdot(X, F)$ be the fiberwise de Rham cohomology group of F, which is a vector bundle over S equipped with the Gauß-Manin flat connection $\nabla^{H^\cdot(X,F)}$. Let $g^{H^\cdot(X,F)}$ be the metric on $H^\cdot(X, F)$ induced by the L^2-metric on $\Omega^\cdot(X, F)$ via the Hodge theorem.

As in §2.0.2, we denote by

$$\begin{aligned} & f\big(F, \nabla^F, g^F\big) \in \Omega^{\mathrm{odd}}(M) \;, \\ & f\big(H^\cdot(X, F), \nabla^{H^\cdot(X,F)}, g^{H^\cdot(X,F)}\big) \in \Omega^{\mathrm{odd}}(S) \end{aligned} \tag{2.0.27}$$

the Bismut-Lott odd characteristic forms associated with $f(x) = x \exp(x^2)$. Let

$$\begin{aligned} & f\big(F, \nabla^F\big) \in H^{\mathrm{odd}}(M, \mathbb{R}) \;, \\ & f\big(H^\cdot(X, F), \nabla^{H^\cdot(X,F)}\big) \in H^{\mathrm{odd}}(S, \mathbb{R}) \end{aligned} \tag{2.0.28}$$

be their cohomology classes.

Let $e(TX)$ denote the Euler class of TX. In [BL95, §3], the authors prove the following Riemann-Roch-Grothendieck formula

$$f\big(H^\cdot(X, F), \nabla^{H^\cdot(X,F)}\big) = \pi_*\big[e(TX) f\big(F, \nabla^F\big)\big] \in H^{\mathrm{odd}}(S, \mathbb{R}) \;. \tag{2.0.29}$$

They also construct an analytic torsion form

$$\mathscr{T}(T^H M, g^{TX}, g^F) \in \Omega^{\mathrm{even}}(S) \tag{2.0.30}$$

satisfying

$$\begin{aligned} & d_S \mathscr{T}(T^H M, g^{TX}, g^F) \\ & = \pi_*\big[e(TX, \nabla^{TX}) f\big(F, \nabla^F, g^F\big)\big] - f\big(H^\cdot(X, F), \nabla^{H^\cdot(X,F)}, g^{H^\cdot(X,F)}\big) \;. \end{aligned} \tag{2.0.31}$$

Moreover, they show that the degree zero component of $\mathscr{T}(T^H M, g^{TX}, g^F)$ is the Ray-Singer analytic torsion [RS71] associated with (X, g^{TX}, F, g^F).

In this article, we extend these constructions to the following setting. Recalling that the flat fibration $q : \mathcal{N} \to M$ is defined in §2.0.2, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{N} \arrow[d, "q"'] \arrow[dr, "r"] & \\ M \arrow[r, "\pi"'] & S \;. \end{tikzcd} \tag{2.0.32}$$

Let Y be the fiber of $r : \mathcal{N} \to S$. Put

$$\mathscr{F} = \Omega^\cdot(X, \mathscr{E}) \;. \tag{2.0.33}$$

Let $d_X^{\mathscr{E}}$ be the lift of the de Rham operator on $\Omega^\cdot(X)$ to $\mathscr{F}$. We will extend Bismut-Lott's constructions to the family of de Rham Dolbeault complexes $\big(\mathscr{F}, \overline{\partial}_N^E + d_X^{\mathscr{E}}\big)$ over S.

For certain reasons, we make the following simplifications. Let L be a complex line bundle over $\mathcal{N}$ constructed in the same way as E. Let g^L be a Hermitian metric on L. We assume that

$$c_1(L,g^L)\big|_N \in \Omega^{1,1}(N) := \mathscr{C}^\infty(N, T^*N\otimes\overline{T^*N}) \tag{2.0.34}$$

is positive. Put

$$E_p = E\otimes L^p \ . \tag{2.0.35}$$

We replace E by E_p with p large enough. By Kodaira's vanishing theorem, we have $H^{>0}(N,E_p)=0$. Put

$$H_p = H^0(N,E_p) \ , \tag{2.0.36}$$

which is a flat vector bundle over M. Let $\mathscr{E}_p$ (resp. $\mathscr{F}_p$) be $\mathscr{E}$ (resp. $\mathscr{F}$) with E replaced by E_p. An argument using the Leray spectral sequence yields

$$H^\cdot\big(\mathscr{F}_p, \overline{\partial}^E_N + d^{\mathscr{E}}_X\big) = H^\cdot(X,H_p) \ . \tag{2.0.37}$$

We also assume that

$$\big(g^L\big)^{-1} d_X g^L \in \mathscr{C}^\infty(\mathcal{N}, T^*X) \tag{2.0.38}$$

is nowhere-zero. This assumption implies that for $p\gg 1$,

$$H^\cdot\big(\mathscr{F}_p, \overline{\partial}^{E_p}_N + d^{\mathscr{E}_p}_X\big) = H^\cdot(X,H_p) = 0 \ . \tag{2.0.39}$$

Applying (2.0.29) with F replaced by H_p and comparing with (2.0.39), we get

$$\pi_*\big[e(TX)f\big(H_p,\nabla^{H_p}\big)\big] = 0 \in H^{\mathrm{odd}}(S,\mathbb{R}) \ . \tag{2.0.40}$$

Let $\alpha_{p,t}\in\Omega^{\mathrm{odd}}(M)$ be the α_t defined by (2.0.20) with E replaced by E_p. By (2.0.23) and (2.0.40), the differential form

$$\pi_*\big[e(TX,\nabla^{TX})\alpha_{p,t}\big] \in \Omega^{\mathrm{odd}}(S) \tag{2.0.41}$$

is exact. Following the same procedure in §2.0.3, we construct an analytic torsion form

$$\mathscr{T}_{\mathrm{tot},t}\big(T^HM, g^{TN}, g^{TX}, g^{E_p}\big) \in \Omega^{\mathrm{even}}(S) \tag{2.0.42}$$

satisfying

$$d_S\mathscr{T}_{\mathrm{tot},t}\big(T^HM, g^{TN}, g^{TX}, g^{E_p}\big) = \pi_*\big[e(TX,\nabla^{TX})\alpha_{p,t}\big] \ . \tag{2.0.43}$$

Let

$$\mathscr{T}(g^{TN}, g^{E_p}) \in \Omega^{\mathrm{even}}(M) \tag{2.0.44}$$

be the analytic torsion form defined in §2.0.3 with E replaced by E_p.

Let g^{H_p} be the metric on H_p induced by the L^2-metric on $\mathscr{E}_p$ via the Hodge theorem. Let

$$\mathscr{T}\big(T^HM, g^{TX}, g^{H_p}\big) \in \Omega^{\mathrm{even}}(S) \tag{2.0.45}$$

be the Bismut-Lott analytic torsion form with (F,g^F) replaced by $\big(H_p,g^{H_p}\big)$.

Theorem 2.0.4. *For p large enough, we have*

$$\begin{aligned}\lim_{t\to\infty} \mathscr{T}_{\mathrm{tot},t}\big(T^H M, g^{TN}, g^{TX}, g^{E_p}\big) \\ = \mathscr{T}\big(T^H M, g^{TX}, g^{H_p}\big) \,.\end{aligned} \tag{2.0.46}$$

For p large enough, modulo exact forms, we have

$$\begin{aligned}&\lim_{t\to 0} \mathscr{T}_{\mathrm{tot},t}\big(T^H M, g^{TN}, g^{TX}, g^{E_p}\big) \\ &= \mathscr{T}\big(T^H M, g^{TX}, g^{H_p}\big) + \pi_*\Big[e(TX, \nabla^{TX})\mathscr{T}(g^{TN}, g^{E_p})\Big] \,.\end{aligned} \tag{2.0.47}$$

Moreover, if $\dim X$ *is odd, for $t > 0$, we have the identity modulo exact forms*

$$\mathscr{T}_{\mathrm{tot},t}\big(T^H M, g^{TN}, g^{TX}, g^{E_p}\big) = \mathscr{T}\big(T^H M, g^{TX}, g^{H_p}\big) \,. \tag{2.0.48}$$

This article is organized as follows.

In §2.1, we recall some standard constructions and known results. Most of them can be found in [BerGV04] and [BL95, §1].

In §2.2, we construct characteristic classes for flat fibrations and prove Theorem 2.0.1.

In §2.3, we prove Theorem 2.0.3. As a consequence, we establish Theorem 2.0.2. We also construct the analytic torsion form $\mathscr{T}(g^{TN}, g^E)$.

In §2.4, we construct the analytic torsion form $\mathscr{T}_{\mathrm{tot},t}\big(T^H M, g^{TN}, g^{TX}, g^{E_p}\big)$. We also state several intermediate theorems and show that these theorems imply Theorem 2.0.4.

In §2.5, we prove the intermediate theorems stated in §2.4.

The results in §2.2 and §2.3 were announced in [Zh16].

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2.1. **Preliminaries.**

The results in this section can be found in [B86, §1], [BL95, §1], [BerGV04, §1].

2.1.1. *Superalgebras.*

In the sequel, the algebras will be over $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. A superalgebra is an algebra A equipped with a $\mathbb{Z}_2$-grading $A = A^+ \oplus A^-$, such that

$$A^+A^\pm \subseteq A^\pm \,, \quad A^-A^\pm \subseteq A^\mp \,. \tag{2.1.1}$$

Let A be a superalgebra. An element $a \in A$ is said to be homogeneous if $a \in A^\pm$. We denote $\deg a = 0$ (resp. $\deg a = 1$) if $a \in A^+$ (resp. $a \in A^-$).

The supercommutator of two homogeneous elements $a, b \in A$ is defined by

$$[a, b] = ab - (-1)^{\deg a \deg b} ba \,. \tag{2.1.2}$$

Also $[\cdot,\cdot]$ extends by linearity to the whole algebra A.

Definition 2.1.2. Let A and B be two superalgebras. The $\mathbb{Z}_2$-graded tensor product $A\widehat{\otimes}B$ is identified with $A \otimes B$ as vector spaces, and the multiplication is given by

$$(a_1 \otimes b_2) \cdot (a_2 \otimes b_2) = (-1)^{\deg a_2 \deg b_1} a_1 a_2 \otimes b_1 b_2 \ , \tag{2.1.3}$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$ are homogeneous elements.

Definition 2.1.3. Let A be a superalgebra. A super A-module is a $\mathbb{Z}_2$-graded vector space $V = V^+ \oplus V^-$ equipped with an action of A, such that

$$A^+ V^\pm \subseteq A^\pm \ , \quad A^- V^\pm \subseteq A^\mp \ . \tag{2.1.4}$$

Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}_2$-graded vector space. Set

$$\tau = \mathrm{id}_{V^+} - \mathrm{id}_{V^-} \in \mathrm{End}(V) \ , \tag{2.1.5}$$

and

$$\mathrm{End}^\pm(V) = \Big\{ a \in \mathrm{End}(V) \ : \ \tau a = \pm a \tau \Big\} \ . \tag{2.1.6}$$

Then $\mathrm{End}(V) = \mathrm{End}^+(V) \oplus \mathrm{End}^-(V)$ is a superalgebra, and V is a super $\mathrm{End}(V)$-module.

For $a \in \mathrm{End}(V)$, its supertrace is defined by

$$\mathrm{Tr_s}\big[a\big] = \mathrm{Tr}\big[\tau a\big] \ . \tag{2.1.7}$$

For any $a, b \in \mathrm{End}(V)$, we have

$$\mathrm{Tr_s}\big[[a,b]\big] = 0 \ . \tag{2.1.8}$$

In the whole article, we apply the superalgebra language to the following geometric settings.

Let M be a $\mathscr{C}^\infty$-manifold. We denote by $\Omega^\cdot(M)$ be the algebra of differential forms on M. We always equip $\Omega^\cdot(M)$ with the $\mathbb{Z}_2$-grading $\Omega^\cdot(M) = \Omega^{\mathrm{even}}(M) \oplus \Omega^{\mathrm{odd}}(M)$. Then $\Omega^\cdot(M)$ is a supercommutative superalgebra, i.e., $[\alpha_1, \alpha_2] = 0$ for $\alpha_1, \alpha_2 \in \Omega^\cdot(M)$.

Let F be a complex vector bundle over M. We denote by $\Omega^\cdot(M,F)$ the vector space of differential forms on M with values in F. We equip $\Omega^\cdot(M,F)$ with the $\mathbb{Z}_2$-grading $\Omega^{\mathrm{even/odd}}(M,F)$. Then $\Omega^\cdot(M,F)$ is a super $\Omega^\cdot(M)$-module.

2.1.2. *Clifford algebras and their representations.*

Let V be a real vector space. Let g^V be an Euclidean metric on V. Let

$$\bigotimes V := \bigoplus_{j=0}^{\infty} V^{\otimes j} \tag{2.1.9}$$

be the tensor algebra of V.

Definition 2.1.4. Let $I \subseteq \bigotimes V$ be a bi-ideal generated by

$$u \otimes v + v \otimes u + 2 g^V(u,v) \ , \quad u, v \in V \ . \tag{2.1.10}$$

Set

$$C(V, g^V) = \big(\bigotimes V\big)/I \ , \tag{2.1.11}$$

called the Clifford algebra associated with (V, g^V).

We could also define the following algebra

$$\widehat{C}(V, g^V) = C(V, -g^V) \,. \tag{2.1.12}$$

Let

$$c : V \to C(V, g^V) \,, \quad \hat{c} : V \to \widehat{C}(V, g^V) \tag{2.1.13}$$

be the maps induced by the canonical injection $V \to \bigotimes V$. For $u, v \in V$, we have

$$\begin{aligned} c(u)c(v) + c(v)c(u) &= -2g^V(u, v) \,, \\ \hat{c}(u)\hat{c}(v) + \hat{c}(v)\hat{c}(u) &= 2g^V(u, v) \,. \end{aligned} \tag{2.1.14}$$

Let $e_1, \cdots, e_n \in V$ be an orthogonal basis of V. Then

$$c(e_{j_1})c(e_{j_2}) \cdots c(e_{j_r}) \,, \quad 0 \leqslant r \leqslant n \,,\ j_1 < j_2 < \cdots < j_r \,, \tag{2.1.15}$$

is a basis of $C(V, g^V)$,

$$\hat{c}(e_{j_1})\hat{c}(e_{j_2}) \cdots \hat{c}(e_{j_r}) \,, \quad 0 \leqslant r \leqslant n \,,\ j_1 < j_2 < \cdots < j_r \,, \tag{2.1.16}$$

is a basis of $\widehat{C}(V, g^V)$.

The algebras $C(V, g^V)$ $\widehat{C}(V, g^V)$ are superalgebras with $C^{\pm}(V, g^V)$, $\widehat{C}^{\pm}(V, g^V)$ generated by the terms in (2.1.15), (2.1.16) with r even/odd.

For $v \in V$, let $v^* \in V^*$ be its dual (with respect to g^V). Let $v^* \wedge$ be the operator on $\Lambda^\cdot V^*$ sending α to $v^* \wedge \alpha$. Let i_v be the operator on $\Lambda^\cdot V^*$ sending $\alpha(\cdots)$ to $\alpha(v, \cdots)$.

Set

$$\begin{aligned} c \ : \ V &\to \mathrm{End}(\Lambda^\cdot V^*) \\ v &\mapsto v^* \wedge - i_v \,. \end{aligned} \tag{2.1.17}$$

For $u, v \in V$, we have

$$c(u)c(v) + c(v)c(u) + 2g^V(u, v) = 0 \,. \tag{2.1.18}$$

Thus c extends to a representation

$$c \ : \ C(V, g^V) \to \mathrm{End}\big(\Lambda^\cdot V^*\big) \,. \tag{2.1.19}$$

This representation will be referred to as the real representation of the Clifford algebra.

In the same spirit, we can construct

$$\begin{aligned} \hat{c} \ : \ V &\to \mathrm{End}(\Lambda^\cdot V^*) \\ v &\mapsto v^* \wedge + i_v \,, \end{aligned} \tag{2.1.20}$$

which extends to a representation

$$\hat{c} \ : \ \widehat{C}(V, g^V) \to \mathrm{End}\big(\Lambda^\cdot V^*\big) \,. \tag{2.1.21}$$

Now we suppose that V is equipped with a complex structure $J \in \mathrm{End}(V)$ and that g^V is J-invariant, i.e.,

$$g^V(\cdot, \cdot) = g^V(J\cdot, J\cdot) \,. \tag{2.1.22}$$

Set

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \,. \tag{2.1.23}$$

The action of J extends $\mathbb{C}$-linearly to $V_{\mathbb{C}}$. The Euclidean metric g^V extends to a $\mathbb{C}$-bilinear form on $V_{\mathbb{C}}$.

Set

$$V_{\mathbb{C}}^{1,0} = \Big\{ v \in V_{\mathbb{C}} \ : \ Jv = iv \Big\} \ , \tag{2.1.24}$$
$$V_{\mathbb{C}}^{0,1} = \Big\{ v \in V_{\mathbb{C}} \ : \ Jv = -iv \Big\} \ .$$

We have

$$V_{\mathbb{C}} = V_{\mathbb{C}}^{1,0} \oplus V_{\mathbb{C}}^{0,1} \ . \tag{2.1.25}$$

For $v \in \mathbb{V}_{\mathbb{C}}$, let $v^{(1,0)}$ (resp. $v^{(0,1)}$) be its component in $V_{\mathbb{C}}^{1,0}$ (resp. $V_{\mathbb{C}}^{0,1}$).

Let $V_{\mathbb{C}}^*$ be the vector space of $\mathbb{R}$-linear forms on $V_{\mathbb{C}}$. For $v \in V_{\mathbb{C}}$, let $v^* \in V_{\mathbb{C}}^*$ be its dual (with respect to g^V).

Set

$$V_{\mathbb{C}}^{*,1,0} = \Big\{ f \in V_{\mathbb{C}}^* \ : \ f \circ J = if \Big\} \ , \tag{2.1.26}$$
$$V_{\mathbb{C}}^{*,0,1} = \Big\{ f \in V_{\mathbb{C}}^* \ : \ f \circ J = -if \Big\} \ .$$

For $v \in V_{\mathbb{C}}^{1,0}$ (resp. $v \in V_{\mathbb{C}}^{0,1}$), we have $v^* \in V_{\mathbb{C}}^{*,0,1}$ (resp. $v^* \in V_{\mathbb{C}}^{*,1,0}$).

Set

$$\begin{aligned} c : V &\to \mathrm{End}\big(\Lambda^{\cdot}(V_{\mathbb{C}}^{*,0,1})\big) \\ v &\mapsto v^{(1,0),*} \wedge - i_{v^{(0,1)}} \ . \end{aligned} \tag{2.1.27}$$

For $u, v \in V$, we have

$$c(u)c(v) + c(v)c(u) + g^V(u,v) = 0 \ . \tag{2.1.28}$$

Thus c extends to a representation

$$c \ : \ C\big(V, \frac{1}{2} g^V\big) \to \mathrm{End}\big(\Lambda^{\cdot}(V_{\mathbb{C}}^{*,0,1})\big) \ . \tag{2.1.29}$$

This representation will be referred to as the complex representation.

2.1.3. *Even/odd characteristic classes.*

Let M be a $\mathscr{C}^\infty$-manifold. Let F be a complex vector bundle over M of rank r.

Let ∇^F be a connection on F. Then ∇^F induces a differential operator

$$\nabla^F : \Omega^{\cdot}(M, F) \to \Omega^{\cdot+1}(M, F) \ . \tag{2.1.30}$$

Let

$$\nabla^{F,2} \in \Omega^2(M, \mathrm{End}(F)) \tag{2.1.31}$$

be the curvature of ∇^F.

For $\omega \in \Omega^k(M)$, put

$$\varphi\omega = (2\pi i)^{-k/2} \omega \ . \tag{2.1.32}$$

Let $\mathrm{Tr}\big[\,\cdot\,\big] : \mathrm{End}(F) \to \mathbb{C}$ be the trace map, which extends to

$$\mathrm{Tr}\big[\,\cdot\,\big] : \Omega^{\cdot}(M, \mathrm{End}(F)) \to \Omega^{\cdot}(M) \tag{2.1.33}$$

such that for $\alpha \in \Omega^{\cdot}(M)$, $A \in \mathscr{C}^\infty(M, \mathrm{End}(F))$,

$$\mathrm{Tr}\big[\omega A\big] = \omega \, \mathrm{Tr}\big[A\big]. \tag{2.1.34}$$

Let $f \in \mathbb{C}[Z]$ be a polynomial.

The following theorem plays a central role in the classical Chern-Weil theory.

Theorem 2.1.5 (Chern-Weil). *The differential form*

$$\varphi \operatorname{Tr}\left[f(-\nabla^{F,2})\right] \in \Omega^{\mathrm{even}}(M) \tag{2.1.35}$$

is real and closed. The cohomology class

$$f(F) := \left[\varphi \operatorname{Tr}\left[f(-\nabla^{F,2})\right]\right] \in H^{\mathrm{even}}(M) \tag{2.1.36}$$

does not depend on the choice of ∇^F.

Now we assume that ∇^F is a flat connection, i.e.,

$$\nabla^{F,2} = 0 \ . \tag{2.1.37}$$

Then

$$\varphi \operatorname{Tr}\left[f(-\nabla^{F,2})\right] = f(0)r \ . \tag{2.1.38}$$

For flat vector bundles, there are non trivial characteristic classes of odd degree. We will follow the construction of Bismut-Lott [BL95, §1].

Let g^F be a Hermitian metric on F. Let $\nabla^{F,*}$ be the adjoint connection, i.e., for $\sigma_1, \sigma_2 \in \mathscr{C}^\infty(M,F)$ and $U \in \mathscr{C}^\infty(M,TM)$, we have

$$g^F(\nabla^F_U \sigma_1, \sigma_2) + g^F(\sigma_1, \nabla^{F,*}_U \sigma_2) = U g^F(\sigma_1, \sigma_2) \ . \tag{2.1.39}$$

Then

$$\nabla^{F,*,2} = 0 \ , \tag{2.1.40}$$

i.e., $\nabla^{F,*}$ is also a flat connection.

Set

$$\omega^F = \nabla^{F,*} - \nabla^F \in \Omega^1(M, \operatorname{End}(F)) \ . \tag{2.1.41}$$

Let $f \in \mathbb{C}[Z]$ be an odd polynomial.

Set

$$f(F, \nabla^F, g^F) = \sqrt{2\pi i}\varphi \operatorname{Tr}\left[f(\omega^F/2)\right] \in \Omega^{\mathrm{odd}}(M) \ . \tag{2.1.42}$$

The following theorem is established by Bismut-Lott [BL95, Theorem 1.8].

Theorem 2.1.6. *The differential form*

$$f(F, \nabla^F, g^F) \in \Omega^{\mathrm{odd}}(M) \tag{2.1.43}$$

is real and closed. The cohomology class

$$f(F, \nabla^F) := \left[f(F, \nabla^F, g^F)\right] \in H^{\mathrm{odd}}(M) \tag{2.1.44}$$

does not depend on the choice of g^F.

Remark 2.1.7. If $f \in \mathbb{C}[Z]$ is an even polynomial, by [BL95, Proposition 1.3], we have

$$\operatorname{Tr}\left[f(\omega^F)\right] = f(0)r \ . \tag{2.1.45}$$

2.1.4. *Fibrations equipped with a connection and a fiberwise metric.*

Let $\pi : M \to S$ be a smooth fibration with compact fiber X.

Let TX be the relative tangent bundle of the fibration. We equip the fibration with a connection. Namely let

$$TM = T^H M \oplus TX \tag{2.1.46}$$

be a smooth splitting of TM. Then $T^H M \simeq \pi^* TS$. Let

$$P^{TX} : TM \to TX \ , \quad P^{T^H M} : TM \to T^H M \tag{2.1.47}$$

be the projections with respect to (2.1.46). For $U \in TS$, let $U^H \in T^H M$ be the lift of U, so that $\pi_* U^H = U$.

If U, V are vector fields on S, set

$$T(U,V) = [U,V]^H - [U^H, V^H] \ . \tag{2.1.48}$$

We have $T \in \Omega^2(S, \mathscr{C}^\infty(X, TX))$. We call T the curvature of the fibration.

We equip TX, TS with metrics g^{TX}, g^{TS}. Let $\pi^* g^{TS}$ be the induced metric on $T^H M$. Set

$$g^{TM} = \pi^* g^{TS} \oplus g^{TX} \ , \tag{2.1.49}$$

which is a Riemannian metric on g^{TM}. Let $\langle \cdot, \cdot \rangle$ denote the corresponding scalar product.

Let ∇^{TM} be the Levi-Civita connection on TM associated with $T^H M$ and g^{TM}.

Definition 2.1.8. Let ∇^{TX} be the connection on TX,

$$\nabla^{TX} = P^{TX} \nabla^{TM} P^{TX}. \tag{2.1.50}$$

Then ∇^{TX} does not depend on g^{TS} (cf. [B86, §1(c)]).

Now we give an explicit formula for ∇^{TX}. Let $L_\cdot$ be the Lie derivative. For U a vector field on S, set

$$\omega^{TX}(U) = (g^{TX})^{-1} L_{U^H} g^{TX} \in \mathscr{C}^\infty(M, \mathrm{End}(TX)) \ . \tag{2.1.51}$$

If $V \in TX$, then ∇^{TX}_V coincides with the usual Levi-Civita connection along the fiber X. If $U \in TS$, then (cf. [B86, §1(c)])

$$\nabla^{TX}_{U^H} = L_{U^H} + \frac{1}{2} \omega^{TX}(U) \ . \tag{2.1.52}$$

Put

$$\nabla^{TM,\oplus} = P^{TX} \nabla^{TM} P^{TX} \oplus P^{T^H M} \nabla^{TM} P^{T^H M}. \tag{2.1.53}$$

Definition 2.1.9. For $U \in TM$, set

$$S^{TX}(U) = \nabla^{TM}_U - \nabla^{TM,\oplus}_U \in \mathscr{C}^\infty(M, \mathrm{End}(TM)) \ . \tag{2.1.54}$$

Then $\left\langle S^{TX}(\cdot) \cdot, \cdot \right\rangle$ does not depend on g^{TS} (cf. [B86, §1(c)]).

2.2. The Chern-Weil theory of a flat fibration.

The purpose of this section is to construct certain characteristic classes and characteristic forms on the total space of a flat fibration with compact complex fibers.

This section is organized as follows. In §2.2.1, we state a consequence of the Chern-Weil theory, which will be of constant use in the rest of this section.

In §2.2.2, we define a flat fibration $q : \mathcal{N} \to M$ whose fiber N is a compact complex manifold.

In §2.2.3, we consider a complex vector bundle E over $\mathcal{N}$, which is holomorphic along N and flat along M.

In §2.2.4, we consider certain connections on E. In particular, given a Hermitian metric on E, we construct a unitary connection on E, and we prove that the integral along the fiber of the usual Chern-Weil forms associated with this connection vanish in positive degree.

In §2.2.5, we construct odd characteristic forms for E. These characteristic forms will appear on the right-hand side of the Riemann-Roch-Grothendieck formula, which will be proved in §2.3.

In §2.2.6, we construct a natural multiplication of the odd characteristic forms defined in §2.2.5.

2.2.1. *A consequence of Chern-Weil theory.*

Let N be a smooth compact oriented manifold. Let $\big(\Omega^{\cdot}(N), d_N\big)$ be the de Rham complex of smooth differential forms on N, whose cohomology is denoted by $H^{\cdot}(N)$.

Let V be a finite dimensional real vector space.

We will replace the de Rham complex $\big(\Omega^{\cdot}(N), d_N\big)$ by the twisted de Rham complex $\big(\Omega^{\cdot}(N, \Lambda^{\cdot}(V^*)), d_N\big)$, whose cohomology is equal to $H^{\cdot}(N)\widehat{\otimes}\Lambda^{\cdot}(V^*)$.

Let $\big(\Omega^{\cdot}(N \times V), d_{N\times V}\big)$ be the de Rham complex of $N \times V$. Then $\big(\Omega^{\cdot}(N, \Lambda^{\cdot}(V^*)), d_N\big)$ can be identified with the subcomplex of $\big(\Omega^{\cdot}(N \times V), d_{N\times V}\big)$ that consists of forms which are constant along V.

Let $p : N \times V \to N$ and $q : N \times V \to V$ be the natural projections. Let q_* denote integration along the oriented fiber N. If $\alpha \in \Omega^{\cdot}(V), \beta \in \Omega^{\cdot}(N)$, then

$$q_*[\alpha \wedge \beta] = \alpha \int_N \beta \ , \tag{2.2.1}$$

By restricting q_* to forms which are constant along V, we get a map

$$q_* : \Omega^{\cdot}(N, \Lambda^{\cdot}(V^*)) \to \Lambda^{\cdot}(V^*) \ . \tag{2.2.2}$$

Let E be a complex vector bundle of rank r on N and ∇^E be a connection on E. Its curvature $\nabla^{E,2}$ is a smooth section of $\Lambda^2(T^*N)\otimes\mathrm{End}(E)$. The vector bundle E lifts to the vector bundle p^*E on $N \times V$, and ∇^E lifts to a connection on p^*E, which is still denoted by ∇^E. Let S be a smooth section on N of $V^* \otimes \mathrm{End}(E)$. We can view S as a section of $V^* \otimes \mathrm{End}(E)$ on $N \times V$, which is constant along V. Then $\nabla^E + S$ is also a connection on p^*E. Its curvature $(\nabla^E + S)^2$ is a smooth section of $\Big(\Lambda^{\cdot}(T^*N)\widehat{\otimes}\Lambda^{\cdot}(V^*)\Big)^{\mathrm{even}} \otimes \mathrm{End}(E)$ over $N \times V$, which is constant along V.

The following proposition is a direct consequence of Chern-Weil theory.

Proposition 2.2.1. *For any invariant complex polynomial P on $\mathfrak{gl}(r, \mathbb{C})$,*

$$P\big(-(\nabla^E + S)^2\big) \in \Omega^{\cdot}(N, \Lambda^{\cdot}(V^*)) \tag{2.2.3}$$

is closed. Its cohomology class

$$\big[P\big(-(\nabla^E+S)^2\big)\big]\in H^\cdot(N)\widehat{\otimes}\Lambda^\cdot(V^*) \tag{2.2.4}$$

does not depend on ∇^E *or on* S. *In particular,*

$$\big[P\big(-(\nabla^E+S)^2\big)\big]\in H^\cdot(N)\subseteq H^\cdot(N)\widehat{\otimes}\Lambda^\cdot(V^*)\ . \tag{2.2.5}$$

2.2.2. *A flat complex fibration.*

Let G be a Lie group. Let N be a compact complex manifold of dimension n. We assume that G acts holomorphically on N.

Let M be a real manifold. Let $p:P_G\to M$ be a principal G-bundle equipped with a connection.

Set

$$\mathcal{N}=P_G\times_G N\ . \tag{2.2.6}$$

Let $q:\mathcal{N}\to M$ be the natural projection, which induces a fibration with canonical fiber N.

Let $T_{\mathbb{R}}N$ be the real tangent bundle of N. Set $T_{\mathbb{C}}N=T_{\mathbb{C}}N\otimes_{\mathbb{R}}\mathbb{C}$.

The connection over the principal bundle P_G induces a connection over the fibration $q:\mathcal{N}\to M$, i.e., a splitting

$$T\mathcal{N}=T_{\mathbb{R}}N\oplus T^H\mathcal{N}\ , \tag{2.2.7}$$

with $T^H\mathcal{N}\simeq q^*TM$.

The splitting (2.2.7) induces the following identification

$$\Lambda^\cdot(T^*_{\mathbb{C}}\mathcal{N})=\Lambda^\cdot(T^*_{\mathbb{C}}N)\widehat{\otimes}q^*\Lambda^\cdot(T^*_{\mathbb{C}}M)\ . \tag{2.2.8}$$

Let TN be the holomorphic tangent bundle of N. Using the splitting $T_{\mathbb{C}}N=TN\oplus\overline{TN}$, we get a further splitting

$$\Lambda^\cdot(T^*_{\mathbb{C}}\mathcal{N})=\Lambda^\cdot(T^*N)\widehat{\otimes}\Lambda^\cdot(\overline{T^*N})\widehat{\otimes}q^*\Lambda^\cdot(T^*_{\mathbb{C}}M)\ . \tag{2.2.9}$$

Put

$$\Omega^{(p,q,r)}(\mathcal{N})=\mathscr{C}^\infty\big(\mathcal{N},\Lambda^p(T^*N)\widehat{\otimes}\Lambda^q(\overline{T^*N})\widehat{\otimes}q^*\Lambda^r(T^*_{\mathbb{C}}M)\big)\ . \tag{2.2.10}$$

Then

$$\Omega^k(\mathcal{N})=\sum_{p+q+r=k}\Omega^{(p,q,r)}(\mathcal{N})\ . \tag{2.2.11}$$

In the sequel, we assume that the connection on P_G is flat. Then $q:\mathcal{N}\to M$ is a flat fibration, i.e., $T=0$ (cf. (2.1.48)).

Let d_N be the de Rham operator on $\Omega^\cdot(N)$. Let d_M be the de Rham operator on $\Omega^\cdot(M)$, which lifts to $\Omega^\cdot(\mathcal{N})$ in the following sense : let (f_α) be a basis of TM, let (f^α) be the dual basis of T^*M. then

$$d_M=\sum_\alpha(q^*f^\alpha)\wedge L_{f^H_\alpha}\ . \tag{2.2.12}$$

Let $d_{\mathcal{N}}$ be the de Rham operator on $\mathcal{N}$. Since $T=0$, by [BL95, Proposition 3.4], we have we get

$$d_{\mathcal{N}}=d_N+d_M\ . \tag{2.2.13}$$

Let ∂_N (resp. $\overline{\partial}_N$) be the holomorphic (resp. anti-holomorphic) Dolbeault operator on N. We have

$$d_N = \partial_N + \overline{\partial}_N \ . \tag{2.2.14}$$

By (2.2.13) and (2.2.14), we get

$$d_{\mathcal{N}} = \partial_N + \overline{\partial}_N + d_M \ . \tag{2.2.15}$$

We have the following obvious relations

$$\begin{aligned} & d_M^2 = d_N^2 = \partial_N^2 = \overline{\partial}_N^2 = 0 \ , \\ & \big[d_M, d_N\big] = \big[d_M, \partial_N\big] = \big[d_M, \overline{\partial}_N\big] = \big[d_N, \partial_N\big] = \big[d_N, \overline{\partial}_N\big] = \big[\partial_N, \overline{\partial}_N\big] = 0 \ . \end{aligned} \tag{2.2.16}$$

2.2.3. *A fiberwise holomorphic vector bundle.*

Let E_0 be a holomorphic vector bundle over N of rank r. We assume that the action of G on N lifts to a holomorphic action on E_0.

Set

$$E = P_G \times_G E_0 \ , \tag{2.2.17}$$

which is a complex vector bundle over $\mathcal{N}$. Furthermore, E is holomorphic along N.

Let $\overline{\partial}_N^E$ be the fiberwise holomorphic structure of E. Let d_M^E be the lift of the de Rham operator on M to $\Omega^{\cdot}(\mathcal{N}, E)$. We have

$$\overline{\partial}_N^{E,2} = d_M^{E,2} = \big[\overline{\partial}_N^E, d_M^E\big] = 0 \ . \tag{2.2.18}$$

As before, the operator d_M^E can be viewed as a flat connection on $\Omega^{\cdot}(N, E)$.

2.2.4. *Connections.*

Set

$$A^{E''} = \overline{\partial}_N^E + d_M^E \tag{2.2.19}$$

acting on $\Omega^{\cdot}(\mathcal{N}, E)$.

Then, by (2.2.18), we have

$$\big(A^{E''}\big)^2 = 0 \ . \tag{2.2.20}$$

Let $\overline{E}^*$ be the anti-dual vector bundle to E. When replacing the complex structure of N by the conjugate complex structure, $\overline{E}^*$ enjoys exactly the same properties as E.

We construct $\partial_N^{\overline{E}^*}$ $d_M^{\overline{E}^*}$ and $A^{\overline{E}^{*\prime}}$ in the same way as $\overline{\partial}_N^E$, d_M^E and $A^{E''}$. In particular,

$$A^{\overline{E}^{*\prime}} = \partial_N^{\overline{E}^*} + d_M^{\overline{E}^*} \ . \tag{2.2.21}$$

As in (2.2.20), we have

$$\big(A^{\overline{E}^{*\prime}}\big)^2 = 0 \ . \tag{2.2.22}$$

Moreover, as in (2.2.18), we have

$$\partial_N^{\overline{E}^*,2} = d_M^{\overline{E}^*,2} = \big[\overline{\partial}_N^{\overline{E}^*}, d_M^{\overline{E}^*}\big] = 0 \ . \tag{2.2.23}$$

Let g^E be a Hermitian metric on E. Then g^E defines an isomorphism $g^E : E \to \overline{E}^*$.

Set

$$\partial_N^E = (g^E)^{-1}\partial_N^{\overline{E}^*} g^E \ , \quad d_M^{E,*} = (g^E)^{-1} d_M^{\overline{E}^*} g^E \ , \tag{2.2.24}$$

which are operators acting on $\Omega^{\cdot}(\mathcal{N}, E)$. By (2.2.23), we have

$$\partial_N^{E,2} = d_M^{E,*,2} = \left[\overline{\partial}_N^E, d_M^{E,*}\right] = 0\ . \tag{2.2.25}$$

Set

$$A^{E\prime} = (g^E)^{-1} A^{\overline{E}^{*\prime}} g^E = \partial_N^E + d_M^{E,*}\ . \tag{2.2.26}$$

Then, by (2.2.25), we have

$$\left(A^{E\prime}\right)^2 = 0\ . \tag{2.2.27}$$

Let $N^{\Lambda^{\cdot}(T^*M)}$ be the number operator of $\Lambda^{\cdot}(T^*M)$.

Definition 2.2.2. Set

$$\begin{aligned} A^E &= 2^{-N^{\Lambda^{\cdot}(T^*M)}}\left(A^{E\prime} + A^{E\prime\prime}\right)2^{N^{\Lambda^{\cdot}(T^*M)}}\ , \\ B^E &= 2^{-N^{\Lambda^{\cdot}(T^*M)}}\left(A^{E\prime} - A^{E\prime\prime}\right)2^{N^{\Lambda^{\cdot}(T^*M)}}\ . \end{aligned} \tag{2.2.28}$$

By (2.2.20) and (2.2.27), we have

$$A^{E,2} = 2^{-N^{\Lambda^{\cdot}(T^*M)}}\left[A^{E\prime}, A^{E\prime\prime}\right]2^{N^{\Lambda^{\cdot}(T^*M)}} = -B^{E,2}\ . \tag{2.2.29}$$

Set

$$d_N^E = \partial_N^E + \overline{\partial}_N^E\ , \qquad d_M^{E,\mathrm{u}} = \frac{1}{2}\left(d_M^E + d_M^{E,*}\right)\ . \tag{2.2.30}$$

Then

$$A^E = d_N^E + d_M^{E,\mathrm{u}}\ , \tag{2.2.31}$$

which shows that A^E a Hermitian connection on E over $\mathcal{N}$.

Set

$$\omega^E = d_M^{E,*} - d_M^E = \left(g^E\right)^{-1} d_M^E g^E \in \mathscr{C}^\infty\left(\mathcal{N}, T^*M \otimes_{\mathbb{R}} \mathrm{End}(E)\right)\ . \tag{2.2.32}$$

Then

$$B^E = \partial_N^E - \overline{\partial}_N^E + \frac{1}{2}\omega^E\ , \tag{2.2.33}$$

which shows that $B^E \in \Omega^{\cdot}\left(M, \mathrm{End}(\Omega^{\cdot}(N, E))\right)$.

Proposition 2.2.3. *For any invariant polynomial P on $\mathfrak{gl}(r, \mathbb{C})$, we have*

$$\left(\partial_N - \overline{\partial}_N\right) P\left(-A^{E,2}\right) = 0\ . \tag{2.2.34}$$

Also

$$P\left(-A^{E,2}\right) - P\left(-d_N^{E,2}\right) \in \mathrm{im}\left(\partial_N - \overline{\partial}_N\right)\ . \tag{2.2.35}$$

We have the identity

$$q_*\left[P\left(-A^{E,2}\right)\right] = q_*\left[P\left(-d_N^{E,2}\right)\right]\ , \tag{2.2.36}$$

and this is a locally constant function on M.

Proof. Let $N^{\Lambda^\cdot(\overline{T^*N})}$ be the number operator of $\Lambda^\cdot(\overline{T^*N})$ and let $U=(-1)^{N^{\Lambda^\cdot(\overline{T^*N})}}$.

To establish the first two equations in our proposition, we only need to show that

$$d_N UP\big(-A^{E,2}\big)=0\ , \tag{2.2.37}$$

and

$$UP\big(-A^{E,2}\big)-UP\big(-d_N^{E,2}\big)\in \operatorname{im}\big(d_N\big)\ . \tag{2.2.38}$$

By (2.2.33), we have

$$U^{-1}B^EU=d_N^E+\frac{1}{2}\omega^E\ . \tag{2.2.39}$$

Then, by (2.2.29), we have

$$U^{-1}A^{E,2}U=-U^{-1}B^{E,2}U=-\big(d_N^E+\frac{1}{2}\omega^E\big)^2\ . \tag{2.2.40}$$

We may and we will assume that P is homogeneous. By (2.2.40), we have

$$UP\big(-A^{E,2}\big)=(-1)^{\deg P}P\Big(-\big(d_N^E+\frac{1}{2}\omega^E\big)^2\Big)\ . \tag{2.2.41}$$

Applying Proposition 2.2.1 to the right-hand side of (2.2.41), the form on the right-hand side is d_N^E closed. This completes the proof of the first two equations of our proposition. The last identity is a consequence of the first two. □

For any $t\in\mathbb{R}$, set

$$A_t^E=d_N^E+td_M^E+(1-t)d_M^{E,*}\ . \tag{2.2.42}$$

In particular,

$$A_{1/2}^E=A^E\ . \tag{2.2.43}$$

Set

$$V_t=(2-2t)^{N^{\Lambda^\cdot(T^*N)}}(2t)^{N^{\Lambda^\cdot(\overline{T^*N})}}\ . \tag{2.2.44}$$

Lemma 2.2.4. *For $t\neq 0,1$, we have*

$$A_t^{E,2}=4t(1-t)V_t^{-1}A^{E,2}V_t\ . \tag{2.2.45}$$

Proof. By (2.2.19) and (2.2.26), we have

$$\begin{aligned}2tV_t^{-1}2^{-N^{\Lambda^\cdot(T^*M)}}A^{E\prime\prime}2^{N^{\Lambda^\cdot(T^*M)}}V_t&=\overline{\partial}_N^E+td_M^E\ ,\\(2-2t)V_t^{-1}2^{-N^{\Lambda^\cdot(T^*M)}}A^{E\prime}2^{N^{\Lambda^\cdot(T^*M)}}V_t&=\partial_N^E+(1-t)d_M^{E,*}\ .\end{aligned} \tag{2.2.46}$$

By (2.2.18), (2.2.25), (2.2.29) and (2.2.46), we have

$$\begin{aligned}&4t(1-t)V_t^{-1}A^{E,2}V_t\\&=\Big[(2-2t)V_t^{-1}2^{-N^{\Lambda^\cdot(T^*M)}}A^{E\prime}2^{N^{\Lambda^\cdot(T^*M)}}V_t\,,\,2tV_t^{-1}2^{-N^{\Lambda^\cdot(T^*M)}}A^{E\prime\prime}2^{N^{\Lambda^\cdot(T^*M)}}V_t\Big]\\&=\Big[\partial_N^E+(1-t)d_M^{E,*}\,,\,\overline{\partial}_N^E+td_M^E\Big]\\&=\Big(\partial_N^E+(1-t)d_M^{E,*}+\overline{\partial}_N^E+td_M^E\Big)^2=A_t^{E,2}\ .\end{aligned} \tag{2.2.47}$$

□

Now we will extend Proposition 2.2.3 by also considering the extra parameter t.

Theorem 2.2.5. *For any invariant polynomial P on $\mathfrak{gl}(r,\mathbb{C})$ and $t\in\mathbb{R}$, we have*

$$q_*\big[P\big(-A_t^{E,2}\big)\big]=q_*\big[P\big(-d_N^{E,2}\big)\big]\ , \tag{2.2.48}$$

and this is a constant function.

Proof. Since $q_*\big[P\big(-A_t^{E,2}\big)\big]$ is polynomial on t, it is sufficient to consider the case $t\neq 0,1$.

We may suppose that P is homogeneous.

By (2.2.45), we have the identity of smooth forms on S

$$q_*\big[P\big(-A_t^{E,2}\big)\big]=\big(4t(1-t)\big)^{\deg P}q_*\big[V_t^{-1}P\big(-A^{E,2}\big)\big]\ . \tag{2.2.49}$$

Applying Proposition 2.2.3 to the right-hand side of (2.2.49), we get

$$q_*\big[P\big(-A_t^{E,2}\big)\big]=\big(4t(1-t)\big)^{\deg P}q_*\big[V_t^{-1}P\big(-d_N^{E,2}\big)\big]\ . \tag{2.2.50}$$

Since $P\big(-d_N^{E,2}\big)$ is a $(\deg P,\deg P)$-form on N, we have

$$V_t^{-1}P\big(-d_N^{E,2}\big)=\big(4t(1-t)\big)^{-\deg P}P\big(-d_N^{E,2}\big)\ . \tag{2.2.51}$$

By (2.2.50) and (2.2.51), we get (2.2.48).

By Chern-Weil theory, $q_*\big[P\big(-d_N^{E,2}\big)\big]$ is locally constant along M. □

2.2.5. *The odd characteristic forms.*

In the sequel, we denote $\varphi=(2\pi i)^{-\frac{1}{2}N^{\Lambda^{\cdot}(T^*\mathcal{N})}}$.

Let P be an invariant polynomial on $\mathfrak{gl}(r,\mathbb{C})$.

Definition 2.2.6. For any $t\in\mathbb{R}$, set

$$\widetilde{P}_t\big(E,g^E\big)=\sqrt{2\pi i}\varphi\,\left\langle P'\big(-A_t^{E,2}\big),\frac{\omega^E}{2}\right\rangle\ . \tag{2.2.52}$$

Proposition 2.2.7. *For any $t\in\mathbb{R}$, $q_*\big[\widetilde{P}_t\big(E,g^E\big)\big]$ is a closed odd differential form on M. The cohomology class $\Big[q_*\big[\widetilde{P}_t\big(E,g^E\big)\big]\Big]\in H^{\cdot}(M)$ does not depend on g^E.*

Proof. Since $\widetilde{P}_t\big(E,g^E\big)$ is odd and $\dim_{\mathbb{R}}N=2n$ is even, $q_*\big[\widetilde{P}_t\big(E,g^E\big)\big]$ is odd.

We will now prove that the above forms are closed.

We have (cf. [BerGV04, §1.4])

$$\begin{aligned}\sqrt{2\pi i}\varphi\frac{\partial}{\partial t}P\big(-A_t^{E,2}\big)&=-\sqrt{2\pi i}\varphi\left\langle P'\big(-A_t^{E,2}\big),\big[A_t^E,\frac{\partial}{\partial t}A_t^E\big]\right\rangle\\&=-\sqrt{2\pi i}\varphi\, d_{\mathcal{N}}\left\langle P'\big(-A_t^{E,2}\big),\frac{\partial}{\partial t}A_t^E\right\rangle\\&=-d_{\mathcal{N}}\varphi\left\langle P'\big(-A_t^{E,2}\big),\frac{\partial}{\partial t}A_t^E\right\rangle\ .\end{aligned} \tag{2.2.53}$$

Since

$$\frac{\partial}{\partial t}A_t^E=d_M^E-d_M^{E,*}=-\omega^E\ , \tag{2.2.54}$$

we have

$$\sqrt{2\pi i}\varphi\,\frac{\partial}{\partial t}P\big(-A_t^{E,2}\big)=2d_{\mathcal{N}}\widetilde{P}_t\big(E,g^E\big)\ . \tag{2.2.55}$$

By Proposition 2.2.5, we get

$$\frac{\partial}{\partial t} q_*\big[P\big(-A_t^{E,2}\big)\big] = 0\,. \tag{2.2.56}$$

By (2.2.55) and (2.2.56), we get

$$d_M q_*\big[\widetilde{P}_t\big(E, g^E\big)\big] = q_*\big[d_{\mathcal{N}} \widetilde{P}_t\big(E, g^E\big)\big] = 0\,. \tag{2.2.57}$$

Thus $q_*\big[\widetilde{P}_t\big(E, g^E\big)\big]$ is closed.

The fact that $\Big[q_*\big[\widetilde{P}_t\big(E, g^E\big)\big]\Big] \in H^{\cdot}(M)$ is independent of g^E comes from the functoriality of our construction (cf. [BerGV04, §1.5]). □

Now we study the dependence of $\widetilde{P}_t\big(E, g^E\big)$ on t.

Recall that V_t was defined in (2.2.44).

Proposition 2.2.8. *If P is homogeneous, for any $t \in \mathbb{R}$, we have*

$$\widetilde{P}_t\big(E, g^E\big) = \big(4t(1-t)\big)^{\deg P - 1} V_t^{-1} \widetilde{P}_{\frac{1}{2}}\big(E, g^E\big)\,. \tag{2.2.58}$$

In particular,

$$q_*\big[\widetilde{P}_t\big(E, g^E\big)\big] = \big(4t(1-t)\big)^{\deg P - n - 1} q_*\big[\widetilde{P}_{\frac{1}{2}}\big(E, g^E\big)\big]\,. \tag{2.2.59}$$

Proof. Since (2.2.58) is a rational function of t, it is sufficient to consider the case $t \neq 0, 1$.

By (2.2.45), we have

$$\begin{aligned} \Big\langle P'\big(-A_t^{E,2}\big)\,, \frac{\omega^E}{2}\Big\rangle &= \Big\langle P'\big(-4t(1-t)V_t^{-1} A_{\frac{1}{2}}^{E,2} V_t\big)\,, \frac{\omega^E}{2}\Big\rangle \\ &= \big(4t(1-t)\big)^{\deg P'} V_t^{-1} \Big\langle P'\big(-A_{\frac{1}{2}}^{E,2}\big)\,, \frac{\omega^E}{2}\Big\rangle \\ &= \big(4t(1-t)\big)^{\deg P - 1} V_t^{-1} \Big\langle P'\big(-A_{\frac{1}{2}}^{E,2}\big)\,, \frac{\omega^E}{2}\Big\rangle\,, \end{aligned} \tag{2.2.60}$$

which is equivalent to (2.2.58). □

In the sequel, we use the convention

$$\widetilde{P}\big(E, g^E\big) = \widetilde{P}_{\frac{1}{2}}\big(E, g^E\big)\,. \tag{2.2.61}$$

The following proposition is a refinement of Proposition 2.2.7 at the level of differential forms.

Proposition 2.2.9. *We have*

$$\begin{aligned} &d_{\mathcal{N}} \widetilde{P}\big(E, g^E\big) \\ &= \frac{\sqrt{2\pi i}}{2} \varphi\,\big(\frac{\partial}{\partial t} V_t^{-1}\big)_{t=\frac{1}{2}} \big(\partial_N - \overline{\partial}_N\big) \int_0^1 \Big\langle P'\Big(\big(\partial_N^E - \overline{\partial}_N^E + \frac{s\omega^E}{2}\big)^2\Big), \frac{\omega^E}{2}\Big\rangle\, ds\,. \end{aligned} \tag{2.2.62}$$

In particular, for $p = 0, \cdots, n$, we have

$$\Big\{d_{\mathcal{N}} \widetilde{P}\big(E, g^E\big)\Big\}^{(p,p,\cdot)} = 0\,. \tag{2.2.63}$$

Proof. By (2.2.45), we have

$$\begin{aligned}&\frac{\partial}{\partial t}\Big\{\sqrt{2\pi i}\varphi\, P\big(-A_t^{E,2}\big)\Big\}_{t=\frac{1}{2}}\\ &=\frac{\partial}{\partial t}\Big\{\sqrt{2\pi i}\varphi\,\big(4t(1-t)\big)^{\deg P}V_t^{-1}P\big(-A^{E,2}\big)\Big\}_{t=\frac{1}{2}}\,.\end{aligned}\tag{2.2.64}$$

By (2.2.51) and (2.2.64), we have

$$\begin{aligned}&\frac{\partial}{\partial t}\Big\{\sqrt{2\pi i}\varphi\, P\big(-A_t^{E,2}\big)\Big\}_{t=\frac{1}{2}}\\ &=\frac{\partial}{\partial t}\Big\{\sqrt{2\pi i}\varphi\,\big(4t(1-t)\big)^{\deg P}V_t^{-1}\Big(P\big(-A^{E,2}\big)-P\big(-d_N^{E,2}\big)\Big)\Big\}_{t=\frac{1}{2}}\end{aligned}\tag{2.2.65}$$

By (2.2.40), we have

$$\begin{aligned}&P\big(-A^{E,2}\big)-P\big(-d_N^{E,2}\big)\\ &=U\left(P\Big(\big(d_N^E+\frac{\omega^E}{2}\big)^2\Big)-P\big(d_N^{E,2}\big)\right)\,.\end{aligned}\tag{2.2.66}$$

As a consequence of Proposition 2.2.1 (cf. [BerGV04, §1.5]). , we get

$$\begin{aligned}&P\Big(\big(d_N^E+\frac{\omega^E}{2}\big)^2\Big)-P\big(d_N^{E,2}\big)\\ &=d_N\int_0^1\left\langle P'\Big(\big(d_N^E+\frac{s\omega^E}{2}\big)^2\Big),\frac{\omega^E}{2}\right\rangle ds\,.\end{aligned}\tag{2.2.67}$$

Then

$$\begin{aligned}&U\left(P\Big(\big(d_N^E+\frac{\omega^E}{2}\big)^2\Big)-P\big(d_N^{E,2}\big)\right)\\ &=\big(\partial_N-\overline{\partial}_N\big)\int_0^1\left\langle P'\Big(\big(\partial_N^E-\overline{\partial}_N^E+\frac{s\omega^E}{2}\big)^2\Big),\frac{\omega^E}{2}\right\rangle ds\,.\end{aligned}\tag{2.2.68}$$

By (2.2.55), (2.2.65), (2.2.66) and (2.2.68), we get (2.2.62).
For $p=0,1,\cdots,n$, we have

$$V_t^{-1}|_{\Omega^{(p,p,\cdot)}}=(4t(1-t))^{-p}\,,\tag{2.2.69}$$

whose derivative at $t=\frac{1}{2}$ is zero. This proves (2.2.63). □

2.2.6. *Multiplication of odd characteristic forms.*
Put

$$P\big(E,g^E\big)=\varphi P\big(-A_{\frac{1}{2}}^{E,2}\big)\,.\tag{2.2.70}$$

Proposition 2.2.10. *Let P,Q be two invariant polynomials. The following identity holds*

$$\begin{aligned}\widetilde{PQ}\big(E,g^E\big)&=\widetilde{P}\big(E,g^E\big)\wedge Q\big(E,g^E\big)\\ &\quad+P\big(E,g^E\big)\wedge\widetilde{Q}\big(E,g^E\big)\,.\end{aligned}\tag{2.2.71}$$

Proof. We have

$$\begin{aligned}&\left\langle (PQ)'\big(-A^{E,2}\big), \frac{\omega^E}{2}\right\rangle \\ &= \left\langle P'\big(-A^{E,2}\big), \frac{\omega^E}{2}\right\rangle \wedge Q\big(-A^{E,2}\big) + P\big(-A^{E,2}\big) \wedge \left\langle Q'\big(-A^{E,2}\big), \frac{\omega^E}{2}\right\rangle ,\end{aligned} \tag{2.2.72}$$

which implies (2.2.71). □

We equip $\Omega^{\mathrm{even}}(\mathcal{N}) \times \Omega^{\mathrm{odd}}(\mathcal{N})$ with the structure of commutative ring. The addition is the usual one. If $(\alpha, \tilde{\alpha}), (\beta, \tilde{\beta}) \in \Omega^{\mathrm{even}}(\mathcal{N}) \times \Omega^{\mathrm{odd}}(\mathcal{N})$, put

$$(\alpha, \tilde{\alpha}) \cdot (\beta, \tilde{\beta}) = (\alpha \wedge \beta, \tilde{\alpha} \wedge \beta + \alpha \wedge \tilde{\beta}) , \tag{2.2.73}$$

Let $\big(\mathbb{C}\big[\mathfrak{gl}(r, \mathbb{C})\big]\big)^{\mathrm{GL}(r,\mathbb{C})}$ be the ring of invariant polynomials on $\mathfrak{gl}(r, \mathbb{C})$.

Proposition 2.2.11. *The following map is a ring homomorphism.*

$$\begin{aligned}\big(\mathbb{C}\big[\mathfrak{gl}(r, \mathbb{C})\big]\big)^{\mathrm{GL}(r,\mathbb{C})} &\to \Omega^{\mathrm{even}}(\mathcal{N}) \times \Omega^{\mathrm{odd}}(\mathcal{N}) \\ P &\mapsto \Big(P\big(E, g^E\big), \widetilde{P}\big(E, g^E\big)\Big) .\end{aligned} \tag{2.2.74}$$

Proof. This is a direct consequence of Proposition 2.2.10. □

Let F be another complex vector bundle over $\mathcal{N}$ satisfying the same properties as E. Let r' be the rank of F. Let g^F be a Hermitian metric on F.

Let Q be an invariant polynomial on $\mathfrak{gl}(r', \mathbb{C})$.

Motivated by Proposition 2.2.11, we make the following definition.

Definition 2.2.12. We define

$$\widetilde{P}\big(E, g^E\big) * \widetilde{Q}\big(F, g^F\big) = \widetilde{P}\big(E, g^E\big)Q\big(F, g^F\big) + P\big(E, g^E\big)\widetilde{Q}\big(F, g^F\big) . \tag{2.2.75}$$

Proposition 2.2.13.

$$q_*\big[\widetilde{P}\big(E, g^E\big) * \widetilde{Q}\big(F, g^F\big)\big] \in \Omega^{\mathrm{odd}}(M) \tag{2.2.76}$$

is a closed form whose cohomology is independent of g^E and g^F.

Proof. The same strategy in the proof of Proposition 2.2.7 still works. The key step is the following identity

$$2d_{\mathcal{N}}\widetilde{P}\big(E, g^E\big) * \widetilde{Q}\big(F, g^F\big) = \sqrt{2\pi i}\varphi\frac{\partial}{\partial t}\Big(P(-A_t^{E,2})Q(-A_t^{F,2})\Big)_{t=1/2} . \tag{2.2.77}$$

□

2.3. A Riemann-Roch-Grothendieck formula.

In this section we will obtain a Riemann-Roch-Grothendieck formula, that the express the odd Chern classes associated with the flat vector bundle $H^{\cdot}(N, E)$ in terms of the exotic Chern classes for TN, E that were defined in §2.2.5.

This section is organized as follows.

In §2.3.1, we introduce the flat infinite dimensional vector bundle $\mathscr{E} = \Omega^{(0,\cdot)}(N, E)$.

In §2.3.2, we equip TN with a fiberwise Kähler metric, E with a Hermitian metric.

In §2.3.3, we introduce the Levi-Civita superconnection on $\mathscr{E}$.

In §2.3.4, we define the index bundle, which is the fiberwise Dolbeault cohomology group of E. We also show that the even characteristic form of the index bundle is a locally constant function on M.

In §2.3.5, we construct differential forms α_t, β_t in the same way as [BL95, §3(h)]. We state explicit formulas calculating the asymptotics of α_t, β_t as $t \to \infty$ and $t \to 0$. We prove a Riemann-Roch-Grothendieck formula as a consequence of these asymptotic estimates.

In §2.3.6, we prove the theorem stated in §2.3.5. The techniques applied in the proof were initiated by Bismut-Gillet-Soulé [BGS88c, §1(h)] and Bismut-Köhler [BK92]. The key idea is a Lichnerowicz formula involving additional Grassmannian variables da, $d\bar{a}$. The introduction of these extra variables will allow us to obtain the our R.R.G. formula.

Finally, in §2.3.7, following [BL95, §3(j)], we construct analytic torsion forms on M, that transgress the R.R.G. formula at the level of differential forms.

2.3.1. *A flat superconnection and its dual.*

Set

$$\mathscr{E}^q = \mathscr{C}^\infty(N, \Lambda^q(\overline{T^*N}) \otimes E) , \quad \mathscr{E} = \bigoplus_q \mathscr{E}^q . \tag{2.3.1}$$

Then $\mathscr{E}$ is an infinite dimensional flat vector bundle over M. By (2.2.10), we have the identification

$$\Omega^{\cdot}(M, \mathscr{E}) = \Omega^{(0,\cdot,\cdot)}(\mathcal{N}, E) . \tag{2.3.2}$$

Let $\nabla^{\mathscr{E}}$ be the restriction of d_M^E to $\Omega^{\cdot}(M, \mathscr{E})$. Then $\nabla^{\mathscr{E}}$ is the canonical flat connection on $\mathscr{E}$.

Set

$$A^{\mathscr{E}} = \overline{\partial}_N^E + \nabla^{\mathscr{E}}. \tag{2.3.3}$$

Then $A^{\mathscr{E}}$ is a superconnection on $\mathscr{E}$.

We recall that the operator $A^{E''}$ acting on $\Omega^{\cdot}(\mathcal{N}, E)$ is defined by (2.2.19). We have

$$A^{\mathscr{E}} = A^{E''}\big|_{\Omega^{(0,\cdot,\cdot)}(\mathcal{N},E)}. \tag{2.3.4}$$

Then, by (2.2.20), we have

$$A^{\mathscr{E},2} = 0 , \tag{2.3.5}$$

i.e., $A^{\mathscr{E}}$ is a flat superconnection. Equivalently, we have

$$\overline{\partial}_N^{E,2} = \nabla^{\mathscr{E},2} = \left[\overline{\partial}_N^E, \nabla^{\mathscr{E}}\right] = 0 . \tag{2.3.6}$$

Set

$$\overline{\mathscr{E}}^* = \mathscr{C}^\infty(N, \Lambda^{\cdot}(T^*N) \otimes \Lambda^n(\overline{T^*N}) \otimes \overline{E}^*) . \tag{2.3.7}$$

Then $\overline{\mathscr{E}}^*$ is an infinite dimensional flat vector bundle over M. We have the identification

$$\Omega^{\cdot}(M, \overline{\mathscr{E}}^*) = \Omega^{(\cdot,n,\cdot)}(\mathcal{N}, \overline{E}^*) . \tag{2.3.8}$$

Let $\nabla^{\overline{\mathscr{E}}^*}$ be the restriction of $d_M^{\overline{E}^*}$ to $\Omega^{\cdot}(M, \overline{\mathscr{E}}^*)$. Then $\nabla^{\overline{\mathscr{E}}^*}$ is the flat connection on $\overline{\mathscr{E}}^*$. Set

$$A^{\overline{\mathscr{E}}^*} = \partial_N^{\overline{E}^*} + \nabla^{\overline{\mathscr{E}}^*} , \tag{2.3.9}$$

which acts on $\Omega^{\cdot}(M, \overline{\mathscr{E}}^*)$. Then $A^{\overline{\mathscr{E}}^*}$ is a superconnection on $\overline{\mathscr{E}}^*$.

We recall that the operator $A^{\overline{E}^*\prime}$ acting on $\Omega^{\cdot}(\mathcal{N},\overline{E}^*)$ is defined by (2.2.21). We have

$$A^{\overline{\mathscr{E}}^*}=A^{\overline{E}^*\prime}\big|_{\Omega^{(\cdot,n,\cdot)}(\mathcal{N},\overline{E}^*)}\,. \tag{2.3.10}$$

Then, by (2.2.22), we have

$$A^{\overline{\mathscr{E}}^*,2}=0\,, \tag{2.3.11}$$

i.e., $A^{\overline{\mathscr{E}}^*}$ is a flat superconnection.

Let

$$(\cdot,\cdot)_E:\overline{E}^*\times E\to\mathbb{C} \tag{2.3.12}$$

be the canonical sesquilinear form, which extends to

$$(\cdot,\cdot)_E:\big(\Lambda^p(T^*N)\otimes\Lambda^n(\overline{T^*N})\otimes\overline{E}^*\big)\times\big(\Lambda^q(\overline{T^*N})\otimes E\big)\to\Lambda^{p+q}(T^*N)\otimes\Lambda^n(\overline{T^*N})\,. \tag{2.3.13}$$

We define

$$\begin{aligned}(\cdot,\cdot)_{\mathscr{E}}:\overline{\mathscr{E}}^*\times\mathscr{E}&\to\mathbb{C}\\ (\alpha,\beta)&\mapsto\int_N(\alpha,\beta)_E\,.\end{aligned} \tag{2.3.14}$$

Thus $\overline{\mathscr{E}}^*$ is formally the anti-dual of $\mathscr{E}$. For any $\alpha\in\Omega^{\cdot}(M,\overline{\mathscr{E}}^*)$ and $\beta\in\Omega^{\cdot}(M,\mathscr{E})$, the following relations hold

$$\begin{aligned}(\partial_N^{\overline{E}^*}\alpha,\beta)_{\mathscr{E}}+(-1)^{\deg\alpha}(\alpha,\overline{\partial}_N^E\beta)_{\mathscr{E}}&=0\,,\\ (\nabla^{\overline{\mathscr{E}}^*}\alpha,\beta)_{\mathscr{E}}+(-1)^{\deg\alpha}(\alpha,\nabla^{\mathscr{E}}\beta)_{\mathscr{E}}&=d_M(\alpha,\beta)_{\mathscr{E}}\,.\end{aligned} \tag{2.3.15}$$

By (2.3.3), (2.3.9) and (2.3.15), we get

$$(A^{\overline{\mathscr{E}}^*}\alpha,\beta)_{\mathscr{E}}+(-1)^{\deg\alpha}(\alpha,A^{\mathscr{E}}\beta)_{\mathscr{E}}=d_M(\alpha,\beta)_{\mathscr{E}}\,, \tag{2.3.16}$$

i.e., $A^{\overline{\mathscr{E}}^*}$ is the dual superconnection of $A^{\mathscr{E}}$ in the sense of [BL95, Definition 1.5].

2.3.2. *Hermitian metrics and connections on TN, E.*

From now on, we will assume that N is a Kähler manifold.

Let $J:T_{\mathbb{R}}N\to T_{\mathbb{R}}N$ be the complex structure of N.

Proposition 2.3.1. *There exists a fiberwise Kähler metric g^{TN} on TN, i.e., a Hermitian metric on TN whose restriction to each fiber is a Kähler metric.*

Proof. Let (U_i) be a locally finite open cover of M by open balls. Let $(f_i:U_i\to\mathbb{R})$ be an associated partition of unity.

For each U_i, we have the trivialization $\varphi_i:q^{-1}(U_i)\to N\times U_i$ as flat fibrations. Let $\pi_{N,i}:N\times U_i\to N$, $\pi_{U_i}:N\times U_i\to U_i$ be the canonical projections.

Let g_0^{TN} be a Kähler metric on TN_0.

Set

$$g^{TN}=\sum_i\varphi_i^*\big((\pi_{U_i}^*f_i)(\pi_{N,i}^*g_0^{TN})\big)\,. \tag{2.3.17}$$

Then g^{TN} satisfies the required conditions. ∎

Let g^{TN} be a fiberwise Kähler metric on TN. Let

$$\omega \in \mathscr{C}^\infty\big(\mathcal{N}, T^*N \otimes \overline{T^*N}\big) \tag{2.3.18}$$

be the associated fiberwise Kähler form. Let

$$dv_N = \frac{\omega^n}{n!} \in \mathscr{C}^\infty\big(\mathcal{N}, \Lambda^{2n}(T^*_{\mathbb{R}}N)\big) \tag{2.3.19}$$

be the induced fiberwise volume form.

Let $g^{\overline{TN}}$, $g^{\Lambda^\cdot(\overline{T^*N})}$ be the Hermitian metrics on $\overline{TN}$, $\Lambda^\cdot(\overline{T^*N})$ induced by g^{TN}.

Let $g^{T_{\mathbb{R}}N}$ be the Riemannian metric on $T_{\mathbb{R}}N$ induced by g^{TN}.

Let $\nabla^{T_{\mathbb{R}}N}$ be the conection on $T_{\mathbb{R}}N$ associated with the metric $g^{T_{\mathbb{R}}N}$ and with the horizontal vector bundle $T^H\mathcal{N}$ that was defined in §2.1.4. We recall that the connection A^{TN} on TN is defined by (2.2.28). In the sequel, we change the notation as follows

$$\nabla^{TN} = A^{TN} \ . \tag{2.3.20}$$

Since the metric g^{TN} is fiberwise Kähler, the connection on $T_{\mathbb{R}}N$ induced by ∇^{TN} along the fibre N coincides with $\nabla^{T_{\mathbb{R}}N}$. Moreover the complex structure of $T_{\mathbb{R}}N$ is flat with respect to the flat connection on $\mathcal{N}$. By (2.1.52), (2.2.30), these two connections also coincide in horizontal directions. The conclusion is that the connection $\nabla^{T_{\mathbb{R}}N}$ preserves the complex structure J, and induces the connection ∇^{TN} on TN.

Let $\nabla^{\overline{TN}}$, $\nabla^{\Lambda^\cdot(\overline{T^*N})}$ be the connections on $\overline{TN}$, $\Lambda^\cdot(\overline{T^*N})$ induced by ∇^{TN}.

Let g^E be a Hermitian metric of E. Let ∇^E be the connection on E defined by (2.2.28).

Let $g^{\Lambda^\cdot(T^*_{\mathbb{C}}N)}$ be the $\mathbb{C}$-bilinear form on $\Lambda^\cdot(T^*_{\mathbb{C}}N)$ induced by g^{TN}. Let

$$* : \Lambda^\cdot(T^*_{\mathbb{C}}N) \to \Lambda^{2n-\cdot}(T^*_{\mathbb{C}}N) \tag{2.3.21}$$

be the usual Hodge operator acting on $\Lambda^\cdot(T^*_{\mathbb{C}}N)$, i.e., for $\alpha, \beta \in \Lambda^\cdot(T^*_{\mathbb{C}}N)$,

$$g^{\Lambda^\cdot(T^*_{\mathbb{C}}N)}(\alpha,\beta)dv_N = \alpha \wedge \overline{*\beta} \ .$$

In particular, $*$ maps $\Lambda^\cdot(\overline{T^*N})$ to $\Lambda^n(T^*N) \otimes \Lambda^{n-\cdot}(\overline{T^*N})$.

The Hermitian metric g^E gives a smooth identification $g^E : E \to \overline{E}^*$. The Hodge operator $*$ extends to

$$*^E : \Lambda^\cdot(\overline{T^*N}) \otimes E \to \Lambda^n(T^*N) \otimes \Lambda^{n-\cdot}(\overline{T^*N}) \otimes \overline{E}^* \ . \tag{2.3.22}$$

Let $g^{\mathscr{E}}$ be a Hermitian metric on $\mathscr{E}$, such that for $\alpha, \beta \in \mathscr{E}$,

$$\begin{aligned} g^{\mathscr{E}}(\alpha,\beta) &= \frac{1}{(2\pi)^n}\int_N \big(g^{\Lambda^\cdot(\overline{T^*N})} \otimes g^E\big)(\alpha,\beta)dv_N \\ &= \frac{(-1)^{\deg\alpha\,\deg\beta}}{(2\pi)^n}(*^E\alpha,\beta)_{\mathscr{E}} \ . \end{aligned} \tag{2.3.23}$$

Set

$$\omega^{\mathscr{E}} = \big(g^{\mathscr{E}}\big)^{-1}\nabla^{\overline{\mathscr{E}}^*}g^{\mathscr{E}} \in \mathscr{C}^\infty(M, T^*M \otimes \mathrm{End}(\mathscr{E})) \tag{2.3.24}$$

and

$$k_N = \big(dv_N\big)^{-1}d_M dv_N \in \mathscr{C}^\infty(\mathcal{N}, T^*M) \ . \tag{2.3.25}$$

We define ω^{TN} as in (2.2.32). Let $\omega^{\Lambda^\cdot(\overline{T^*N})}$ be the action of ω^{TN} on $\Lambda^\cdot(\overline{T^*N})$. Then $\omega^{\Lambda^\cdot(\overline{T^*N})}$ is just the horizontal variation of the metric $g^{\Lambda^\cdot(\overline{T^*N})}$ on $\Lambda^\cdot\big(\overline{T^*N}\big)$ with respect to the flat connection. We have

$$\omega^{\mathscr{E}} = \omega^{\Lambda^\cdot(\overline{T^*N})} + \omega^E + k_N \ . \tag{2.3.26}$$

2.3.3. *The Levi-Civita superconnection.*

We recall that $A^{\mathscr{E}}$ and $A^{\overline{\mathscr{E}}^*}$ are defined by (2.3.3) and (2.3.9).

Definition 2.3.2. Set

$$A^{\mathscr{E},*} = (*^E)^{-1} A^{\overline{\mathscr{E}}^*} *^E \,, \tag{2.3.27}$$

which acts on $\Omega^{\cdot}(M,\mathscr{E})$. Then $A^{\mathscr{E},*}$ is the adjoint superconnection of $A^{\mathscr{E}}$ (with respect to $g^{\mathscr{E}}$) in the sense of [BL95, Definition 1.6].

By (2.3.11), we have

$$A^{\mathscr{E},*,2} = 0 \,. \tag{2.3.28}$$

Set

$$\begin{aligned} C^{\mathscr{E}} &= 2^{-N_{\Lambda^{\cdot}(T^*M)}} \big(A^{\mathscr{E},*} + A^{\mathscr{E}}\big) 2^{N_{\Lambda^{\cdot}(T^*M)}} \,, \\ D^{\mathscr{E}} &= 2^{-N_{\Lambda^{\cdot}(T^*M)}} \big(A^{\mathscr{E},*} - A^{\mathscr{E}}\big) 2^{N_{\Lambda^{\cdot}(T^*M)}} \,. \end{aligned} \tag{2.3.29}$$

By (2.3.5) and (2.3.28), we have

$$C^{\mathscr{E},2} = -D^{\mathscr{E},2} = 2^{-N_{\Lambda^{\cdot}(T^*M)}} \big[A^{\mathscr{E}}, A^{\mathscr{E},*}\big] 2^{N_{\Lambda^{\cdot}(T^*M)}} \,, \qquad \big[C^{\mathscr{E}}, D^{\mathscr{E}}\big] = 0 \,. \tag{2.3.30}$$

Let $\overline{\partial}_N^{E,*}$ be the formal adjoint of $\overline{\partial}_N^E$ with respect to $g^{\mathscr{E}}$. Set

$$D_N^E = \overline{\partial}_N^E + \overline{\partial}_N^{E,*} \tag{2.3.31}$$

acting on $\mathscr{E}$. Then D_N^E is the fiberwise spin^c-Dirac operator associated to $g^{TN}/2$.

We recall that $\nabla^{\mathscr{E}}$ is defined in §2.3.1. Let $\nabla^{\mathscr{E},*}$ be the adjoint connection. Then

$$\nabla^{\mathscr{E},*} = \nabla^{\mathscr{E}} + \omega^{\mathscr{E}} \,. \tag{2.3.32}$$

Set

$$\nabla^{\mathscr{E},\mathrm{u}} = \frac{1}{2}\big(\nabla^{\mathscr{E},*} + \nabla^{\mathscr{E}}\big) = \nabla^{\mathscr{E}} + \frac{1}{2}\omega^{\mathscr{E}} \,, \tag{2.3.33}$$

which is a unitary connection on $\mathscr{E}$.

We have

$$C^{\mathscr{E}} = D_N^E + \nabla^{\mathscr{E},\mathrm{u}} \,, \quad D^{\mathscr{E}} = \overline{\partial}_N^{E,*} - \overline{\partial}_N^E + \frac{1}{2}\omega^{\mathscr{E}} \,. \tag{2.3.34}$$

Recall that the Levi-Civita superconnection was introduced in [B86].

Proposition 2.3.3. *The superconnection $C^{\mathscr{E}}$ is the Levi-Civita superconnection with respect to $T^H\mathcal{N}$, $g^{T_{\mathbb{R}}N}$ and g^E.*

Proof. Since the metric g^{TN} is fibrewise Kähler, up to the constant $\sqrt{2}$, the operator D_N^E is a standard Dirac operator along the fiber N. As we saw before, the connection $\nabla^{T_{\mathbb{R}}N}$ induced by ∇^{TN} is exactly the connection that was considered in [B86]. Finally, since our fibration is flat, the term in the Levi-Civita superconnection that contains the curvature of our fibration vanishes identically. This completes the proof of our proposition. □

Given $t > 0$, let $C_t^{\mathscr{E}}, D_t^{\mathscr{E}}$ be the objects defined before which are associated with the metrics $g^{TN}/t, g^E$. By (2.3.34), we have

$$C_t^{\mathscr{E}} = t\overline{\partial}_N^{E,*} + \overline{\partial}_N^E + \nabla^{\mathscr{E},\mathrm{u}} \,, \quad D_t^{\mathscr{E}} = t\overline{\partial}_N^{E,*} - \overline{\partial}_N^E + \frac{1}{2}\omega^{\mathscr{E}} \,. \tag{2.3.35}$$

2.3.4. *The index bundle and its characteristic classes.*

Let $H^{\cdot}(N,E_0)$ be the Dolbeault cohomology of E_0. The action of G on E_0 induces an action of G on $H^{\cdot}(N,E_0)$. Set

$$H^{\cdot}(N,E) = P_G \times_G H^{\cdot}(N,E_0)\ . \tag{2.3.36}$$

Let $\nabla^{H^{\cdot}(N,E)}$ be the flat connection on $H^{\cdot}(N,E)$ induced by the flat connection on P_G. For $s \in \mathscr{C}^\infty(M,\mathscr{E})$ satisfying $\overline{\partial}^E_N s = 0$, let $[s]$ denote the corresponding fiberwise Dolbeault cohomology class. Then

$$\nabla^{H^{\cdot}(N,E)}[s] = [\nabla^{\mathscr{E}} s] \in \Omega^1(M, H^{\cdot}(N,E))\ . \tag{2.3.37}$$

By Hodge theory, there is a natural identification

$$H^{\cdot}(N,E) \simeq \ker D^E_N \subseteq \mathscr{E}\ . \tag{2.3.38}$$

Let $g^{H^{\cdot}(N,E)}$ be the metric on $H^{\cdot}(N,E)$ induced by $g^{\mathscr{E}}$ via the above identification.

Let $\nabla^{H^{\cdot}(N,E),*}$ be the adjoint connection of $\nabla^{H^{\cdot}(N,E)}$ with respect to $g^{H^{\cdot}(N,E)}$. Set

$$\begin{aligned} \nabla^{H^{\cdot}(N,E),\mathrm{u}} &= \frac{1}{2}\big(\nabla^{H^{\cdot}(N,E),*} + \nabla^{H^{\cdot}(N,E)}\big)\ ,\\ \omega^{H^{\cdot}(N,E)} &= \nabla^{H^{\cdot}(N,E),*} - \nabla^{H^{\cdot}(N,E)}\ . \end{aligned} \tag{2.3.39}$$

Then $\nabla^{H^{\cdot}(N,E),\mathrm{u}}$ is a unitary connection and $\omega^{H^{\cdot}(N,E)} \in \mathscr{C}^\infty\big(M, \mathrm{End}(H^{\cdot}(N,E))\big)$.

Put

$$\chi(N,E) = \sum_p (-1)^p \dim H^p(N,E)\ . \tag{2.3.40}$$

Proposition 2.3.4. *For any $t>0$, we have*

$$\varphi\,\mathrm{Tr_s}\big[\exp(D^{\mathscr{E},2}_t)\big] = \chi(N,E)\ . \tag{2.3.41}$$

Proof. By the local families index theorem [B86], as $t \to 0$,

$$\varphi\,\mathrm{Tr_s}\big[\exp(D^{\mathscr{E},2}_t)\big] = q_*\big[\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\big] + \mathscr{O}(\sqrt{t})\ . \tag{2.3.42}$$

Furthermore,

$$\begin{aligned} \frac{\partial}{\partial t}\mathrm{Tr_s}\big[\exp(D^{\mathscr{E},2}_t)\big] &= \mathrm{Tr_s}\big[\big[D^{\mathscr{E}}_t, \frac{\partial}{\partial t}D^{\mathscr{E}}_t\big]\exp(D^{\mathscr{E},2}_t)\big]\\ &= \mathrm{Tr_s}\big[\big[D^{\mathscr{E}}_t, (\frac{\partial}{\partial t}D^{\mathscr{E}}_t)\exp(D^{\mathscr{E},2}_t)\big]\big] = 0\ . \end{aligned} \tag{2.3.43}$$

By Proposition 2.2.5 and by the Riemann-Roch-Hirzebruch formula, we have

$$q_*\big[\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\big] = \chi(N,E)\ . \tag{2.3.44}$$

Then (2.3.41) follows from (2.3.42)-(2.3.44). □

2.3.5. *A Riemann-Roch-Grothendieck formula.*

For $t>0$, set

$$\begin{aligned} \alpha_t &= \sqrt{2\pi i}\varphi\,\mathrm{Tr_s}\Big[D^{\mathscr{E}}_t\exp(D^{\mathscr{E},2}_t)\Big]\ ,\\ \beta_t &= \varphi\,\mathrm{Tr_s}\Big[\frac{N^{\Lambda^{\cdot}(\overline{T^*N})}}{2}(1+2D^{\mathscr{E},2}_t)\exp(D^{\mathscr{E},2}_t)\Big]\ . \end{aligned} \tag{2.3.45}$$

Proposition 2.3.5. *For $t>0$, α_t is a closed odd real form on M, whose cohomology class does not depend on the metrics g^{TN}, g^E or on t.*

Proof. By (2.3.30), we have

$$d_M \sqrt{2\pi i}\varphi \operatorname{Tr_s}\left[D_t^{\mathscr{E}} \exp(D_t^{\mathscr{E},2})\right] = \varphi \operatorname{Tr_s}\left[\left[C_t^{\mathscr{E}}, D_t^{\mathscr{E}} \exp(D_t^{\mathscr{E},2})\right]\right] = 0 \,, \tag{2.3.46}$$

which proves the closeness.

Then, by the functoriality of our constructions, $[\alpha_t] \in H^{\cdot}(M)$ does not depend on the metric. □

Proposition 2.3.6. *For any $t > 0$, the following identity holds:*

$$\frac{\partial}{\partial t}\alpha_t = \frac{1}{t} d_M \beta_t \,. \tag{2.3.47}$$

Proof. Set

$$\mathscr{N}_+ = \mathscr{N} \times \mathbb{R}_+ \,, \qquad M_+ = M \times \mathbb{R}_+ \,. \tag{2.3.48}$$

Let

$$q_+ = q \oplus \mathrm{id}_{\mathbb{R}_+} : \mathscr{N}_+ \to M_+ \tag{2.3.49}$$

be the obvious projection. Let t be the coordinate on $\mathbb{R}_+$.

We equip TN with the metric $\frac{1}{t} g^{TN}$. Let $\mathscr{E}_+$, $\omega^{\mathscr{E}_+}$, $C^{\mathscr{E}_+}$, $D^{\mathscr{E}_+}$ be the corresponding objects associated to the new fibration. Then the following identities hold (cf. (2.3.24))

$$\begin{aligned} d_{M_+} &= d_M + dt \wedge \frac{\partial}{\partial t} \,, \\ \omega^{\mathscr{E}_+} &= \omega^{\mathscr{E}} + \frac{1}{t} dt \wedge \left(N^{\Lambda^{\cdot}(\overline{T^*N})} - n\right) . \end{aligned} \tag{2.3.50}$$

Then, by (2.3.34) and (2.3.35), we get

$$\begin{aligned} C^{\mathscr{E}_+} &= C_t^{\mathscr{E}} + dt \wedge \frac{\partial}{\partial t} + \frac{1}{2t} dt \wedge \left(N^{\Lambda^{\cdot}(\overline{T^*N})} - n\right) , \\ D^{\mathscr{E}_+} &= D_t^{\mathscr{E}} + \frac{1}{2t} dt \wedge \left(N^{\Lambda^{\cdot}(\overline{T^*N})} - n\right) . \end{aligned} \tag{2.3.51}$$

Thus

$$\begin{aligned} &\sqrt{2\pi i}\varphi \operatorname{Tr_s}\left[D^{\mathscr{E}_+} \exp(D^{\mathscr{E}_+,2})\right] \\ &= \sqrt{2\pi i}\varphi \operatorname{Tr_s}\left[D^{\mathscr{E}} \exp(D^{\mathscr{E},2})\right] + \frac{1}{2t} dt \wedge \varphi \operatorname{Tr_s}\left[\left(N^{\Lambda^{\cdot}(\overline{T^*N})} - n\right) \exp(D^{\mathscr{E},2})\right] \\ &\quad + \sqrt{2\pi i}\varphi \operatorname{Tr_s}\left[D^{\mathscr{E}} \exp\left(D^{\mathscr{E},2} + \left[D^{\mathscr{E}}, \frac{1}{2t} dt \wedge N^{\Lambda^{\cdot}(\overline{T^*N})}\right]\right)\right] \\ &= \alpha_t + \frac{1}{2t} dt \wedge \varphi \operatorname{Tr_s}\left[N^{\Lambda^{\cdot}(\overline{T^*N})} \exp(D^{\mathscr{E},2})\right] - \chi(N,E) \frac{n}{2t} dt \\ &\quad + \sqrt{2\pi i}\varphi \operatorname{Tr_s}\left[D^{\mathscr{E}}\left[D^{\mathscr{E}}, \exp\left(D^{\mathscr{E},2} + \frac{1}{2t} dt \wedge N^{\Lambda^{\cdot}(\overline{T^*N})}\right)\right]\right] \\ &= \alpha_t + \frac{1}{2t} dt \wedge \varphi \operatorname{Tr_s}\left[N^{\Lambda^{\cdot}(\overline{T^*N})} \exp(D^{\mathscr{E},2})\right] - \chi(N,E) \frac{n}{2t} dt \\ &\quad + \sqrt{2\pi i}\varphi \operatorname{Tr_s}\left[\left[D^{\mathscr{E}}, D^{\mathscr{E}}\right] \exp\left(D^{\mathscr{E},2} + \frac{1}{2t} dt \wedge N^{\Lambda^{\cdot}(\overline{T^*N})}\right)\right] \\ &= \alpha_t + \frac{1}{2t} dt \wedge \beta_t - \chi(N,E) \frac{n}{2t} dt \in \Omega^{\cdot}(M_+) \,. \end{aligned} \tag{2.3.52}$$

By Proposition 2.3.5, we have

$$d_{M_+} \sqrt{2\pi i}\varphi \operatorname{Tr_s}\left[D^{\mathscr{E}_+} \exp(D^{\mathscr{E}_+,2})\right] = 0 \,. \tag{2.3.53}$$

By the first identity in (2.3.50), (2.3.52), (2.3.53), we get (2.3.47). □

Let $f(x) = xe^{x^2}$.

Following [BL95, Definition 1.7], we define the odd real closed form on M given by

$$f(H^\cdot(N,E),\nabla^{H^\cdot(N,E)},g^{H^\cdot(N,E)}) = \sqrt{2\pi i}\varphi\,\mathrm{Tr_s}\left[f(\omega^{H^\cdot(N,E)}/2)\right] . \tag{2.3.54}$$

Put

$$\chi'(N,E) = \sum_p (-1)^p p \dim H^p(N,E) . \tag{2.3.55}$$

Now we state the central result in this section. Its proof will be delayed to §2.3.6.

Theorem 2.3.7. *As $t \to +\infty$,*

$$\begin{aligned} \alpha_t &= f(H^\cdot(N,E),\nabla^{H^\cdot(N,E)},g^{H^\cdot(N,E)}) + \mathscr{O}\big(\frac{1}{\sqrt{t}}\big) ,\\ \beta_t &= \frac{1}{2}\chi'(N,E) + \mathscr{O}\big(\frac{1}{\sqrt{t}}\big) . \end{aligned} \tag{2.3.56}$$

As $t \to 0$,

$$\begin{aligned} \alpha_t = {} & q_*\Big[\widetilde{\mathrm{Td}}(TN,g^{TN}) * \widetilde{\mathrm{ch}}(E,g^E)\Big] \\ & + \frac{1}{2t}d_M q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big] + \mathscr{O}\big(\sqrt{t}\big) ,\\ \beta_t = {} & -\frac{1}{2}q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big] + \frac{n}{2}\chi(N,E) \\ & - \frac{1}{2t}q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big] + \mathscr{O}\big(\sqrt{t}\big) . \end{aligned} \tag{2.3.57}$$

Remark 2.3.8. By Proposition 2.2.3, we have

$$q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big] \in \mathscr{C}^\infty(M) . \tag{2.3.58}$$

Now we prove the following Riemann-Roch-Grothendieck formula.

Theorem 2.3.9. *We have*

$$\begin{aligned} & \Big[f(H^\cdot(N,E),\nabla^{H^\cdot(N,E)},g^{H^\cdot(N,E)})\Big] \\ & = \Big[q_*\Big[\widetilde{\mathrm{Td}}(TN,g^{TN}) * \widetilde{\mathrm{ch}}(E,g^E)\Big]\Big] \in H^{\mathrm{odd}}(M,\mathbb{R}) . \end{aligned} \tag{2.3.59}$$

Proof. We combine Proposition 2.3.5 and Theorem 2.3.7. □

2.3.6. *Several intermediate results, Lichnerowicz formulas and the proof of Theorem 2.3.7.*

We will now introduce various new odd Grassmann variables in order to be able to compute exactly the asymptotics of certain superconnection forms as $t \to 0$, and also to overcome the divergence of certain expressions. Our methods are closely related to the methods of [BGS88b, BGS88c, BK92], where similar difficulties also appeared.

Let a be an additional complex coordinate, ϵ be an auxiliary odd Grassmann variable. For

$$u, v \in \Big\{1\,,\,da\,,\,d\bar{a}\,,\,dad\bar{a}\,,\,\epsilon\,,\,\epsilon da\,,\,\epsilon d\bar{a}\,,\,\epsilon dad\bar{a}\Big\} \tag{2.3.60}$$

and $\sigma \in \Omega^{\cdot}(M)$, we denote

$$(v \wedge \sigma)^u = \begin{cases} \sigma & \text{if } u = v\ , \\ 0 & \text{else}\ . \end{cases} \tag{2.3.61}$$

Lemma 2.3.10. *The following identity holds*

$$\begin{aligned} &\operatorname{Tr_s}\Big[D^{\mathscr{E}} \exp\big(D^{\mathscr{E},2}\big)\Big] \\ =\ &\operatorname{Tr_s}\Big[\exp\Big(- C^{\mathscr{E},2} - da\, \frac{1}{2}\big(\overline{\partial}^E_N + \overline{\partial}^{E,*}_N\big) \\ &\qquad - d\bar{a}\, \big[\overline{\partial}^E_N + \overline{\partial}^{E,*}_N, \frac{\epsilon}{2}\omega^{\mathscr{E}}\big] + da d\bar{a}\, \frac{\epsilon}{2}\omega^{\mathscr{E}}\Big)\Big]^{\epsilon da d\bar{a}} \\ &+ d_M \operatorname{Tr_s}\Big[\frac{1}{2} N^{\Lambda^{\cdot}(\overline{T^*N})} \exp\big(D^{\mathscr{E},2}\big)\Big]\ . \end{aligned} \tag{2.3.62}$$

Proof. By (2.3.30) and (2.3.34), we have

$$\begin{aligned} \big[N^{\Lambda^{\cdot}(\overline{T^*N})}, C^{\mathscr{E},2}\big] &= -\big[N^{\Lambda^{\cdot}(\overline{T^*N})}, D^{\mathscr{E},2}\big] \\ &= -\big[N^{\Lambda^{\cdot}(\overline{T^*N})}, \big[\overline{\partial}^{E,*}_N - \overline{\partial}^E_N, \frac{1}{2}\omega^{\mathscr{E}}\big]\big] \\ &= \big[\overline{\partial}^E_N + \overline{\partial}^{E,*}_N, \frac{1}{2}\omega^{\mathscr{E}}\big]\ , \end{aligned} \tag{2.3.63}$$

which implies

$$\begin{aligned} &\operatorname{Tr_s}\Big[\exp\Big(- C^{\mathscr{E},2} - da\, \frac{1}{2}\big(\overline{\partial}^E_N + \overline{\partial}^{E,*}_N\big) - d\bar{a}\, \big[\overline{\partial}^E_N + \overline{\partial}^{E,*}_N, \frac{\epsilon}{2}\omega^{\mathscr{E}}\big]\Big)\Big]^{\epsilon da d\bar{a}} \\ =\ &\frac{\partial}{\partial b}\operatorname{Tr_s}\Big[- \frac{1}{2}\big(\overline{\partial}^E_N + \overline{\partial}^{E,*}_N\big) \exp\Big(- C^{\mathscr{E},2} + b\big[\overline{\partial}^E_N + \overline{\partial}^{E,*}_N, \frac{1}{2}\omega^{\mathscr{E}}\big]\Big)\Big]_{b=0} \\ =\ &\frac{\partial}{\partial b}\operatorname{Tr_s}\Big[- \frac{1}{2}\big(\overline{\partial}^E_N + \overline{\partial}^{E,*}_N\big) \exp\Big(- C^{\mathscr{E},2} + b\big[N^{\Lambda^{\cdot}(\overline{T^*N})}, C^{\mathscr{E},2}\big]\Big)\Big]_{b=0} \\ =\ &\frac{\partial}{\partial b}\operatorname{Tr_s}\Big[- \frac{1}{2}\big(\overline{\partial}^E_N + \overline{\partial}^{E,*}_N\big)\big[N^{\Lambda^{\cdot}(\overline{T^*N})}, \exp\big(- C^{\mathscr{E},2}\big)\big]\Big] \\ =\ &\operatorname{Tr_s}\Big[- \frac{1}{2}\big[N^{\Lambda^{\cdot}(\overline{T^*N})}, \overline{\partial}^E_N + \overline{\partial}^{E,*}_N\big] \exp\big(- C^{\mathscr{E},2}\big)\Big] \\ =\ &\operatorname{Tr_s}\Big[\frac{1}{2}\big(\overline{\partial}^{E,*}_N - \overline{\partial}^E_N\big) \exp\big(D^{\mathscr{E},2}\big)\Big]\ . \end{aligned} \tag{2.3.64}$$

Then

$$
\begin{aligned}
&\mathrm{Tr_s}\Big[\exp\big(-C^{\mathscr{E},2}-da\,\frac{1}{2}(\overline{\partial}_N^E+\overline{\partial}_N^{E,*})\\
&\qquad\qquad -d\bar{a}\,\big[\overline{\partial}_N^E+\overline{\partial}_N^{E,*},\frac{\epsilon}{2}\omega^{\mathscr{E}}\big]+dad\bar{a}\,\frac{\epsilon}{2}\omega^{\mathscr{E}}\big)\Big]^{\epsilon dad\bar{a}}\\
=\;&\mathrm{Tr_s}\Big[\exp\big(-C^{\mathscr{E},2}-da\,\frac{1}{2}(\overline{\partial}_N^E+\overline{\partial}_N^{E,*})-d\bar{a}\,\big[\overline{\partial}_N^E+\overline{\partial}_N^{E,*},\frac{\epsilon}{2}\omega^{\mathscr{E}}\big]\big)\Big]^{\epsilon dad\bar{a}}\\
&+\mathrm{Tr_s}\Big[\frac{1}{2}\omega^{\mathscr{E}}\exp\big(D^{\mathscr{E},2}\big)\Big]\\
=\;&\mathrm{Tr_s}\Big[\frac{1}{2}(\overline{\partial}_N^{E,*}-\overline{\partial}_N^E+\omega^{\mathscr{E}})\exp\big(D^{\mathscr{E},2}\big)\Big]\\
=\;&\mathrm{Tr_s}\Big[(\overline{\partial}_N^{E,*}-\overline{\partial}_N^E+\frac{1}{2}\omega^{\mathscr{E}})\exp\big(D^{\mathscr{E},2}\big)\Big]-\mathrm{Tr_s}\Big[\frac{1}{2}(\overline{\partial}_N^{E,*}-\overline{\partial}_N^E)\exp\big(D^{\mathscr{E},2}\big)\Big]\\
=\;&\mathrm{Tr_s}\Big[D^{\mathscr{E}}\exp\big(D^{\mathscr{E},2}\big)\Big]-\mathrm{Tr_s}\Big[\big[C^{\mathscr{E}},\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big]\exp\big(D^{\mathscr{E},2}\big)\Big]\\
=\;&\mathrm{Tr_s}\Big[D^{\mathscr{E}}\exp\big(D^{\mathscr{E},2}\big)\Big]-d_M\,\mathrm{Tr_s}\Big[\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\exp\big(D^{\mathscr{E},2}\big)\Big]\,.
\end{aligned}
\tag{2.3.65}
$$

The last equation is just what we needed to prove. This completes the proof of our proposition. □

Let $\mathcal{N}_+$, M_+, q_+, $\mathscr{E}_+$, $\omega^{\mathscr{E}_+}$, $C^{\mathscr{E}_+}$ and $D^{\mathscr{E}_+}$ be the same as in the proof of Proposition 2.3.6.

Lemma 2.3.11. *Given $t>0$, the following identity holds:*

$$
\begin{aligned}
&\big(N^{\Lambda^{\cdot}(T^*M)}+1+t\frac{\partial}{\partial t}\big)\,\mathrm{Tr_s}\Big[\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\\
=\;&\mathrm{Tr_s}\Big[\exp\big(-C^{\mathscr{E}_+,2}-da\,\frac{1}{2}(\overline{\partial}_N^E+t\overline{\partial}_N^{E,*})\\
&\qquad\qquad -d\bar{a}\,\big[\overline{\partial}_N^E+t\overline{\partial}_N^{E,*},\frac{\epsilon t}{2}\omega^{\mathscr{E}_+}\big]+dad\bar{a}\,\frac{\epsilon t}{2}\omega^{\mathscr{E}_+}\big)\Big]^{\epsilon dad\bar{a}dt}\\
&+\textit{closed form}\,.
\end{aligned}
\tag{2.3.66}
$$

Proof. By (2.3.62), we get

$$
\begin{aligned}
&\mathrm{Tr_s}\Big[D^{\mathscr{E}_+}\exp\big(D^{\mathscr{E}_+,2}\big)\Big]\\
=\;&\mathrm{Tr_s}\Big[\exp\big(-C^{\mathscr{E}_+,2}-da\,\frac{1}{2}(\overline{\partial}_N^E+t\overline{\partial}_N^{E,*})\\
&\qquad\qquad -d\bar{a}\,\big[\overline{\partial}_N^E+t\overline{\partial}_N^{E,*},\frac{\epsilon}{2}\omega^{\mathscr{E}_+}\big]+dad\bar{a}\,\frac{\epsilon}{2}\omega^{\mathscr{E}_+}\big)\Big]^{\epsilon dad\bar{a}}\\
&+d_{M_+}\,\mathrm{Tr_s}\Big[\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\exp\big(D^{\mathscr{E}_+,2}\big)\Big]\,.
\end{aligned}
\tag{2.3.67}
$$

Taking the dt component, we get

$$
\begin{aligned}
&\operatorname{Tr_s}\Big[\frac{1}{2t}\big(N^{\Lambda^{\cdot}(\overline{T^*N})}-n\big)\exp\big(D_t^{\mathscr{E},2}\big)\Big]\\
&\quad+\operatorname{Tr_s}\Big[D_t^{\mathscr{E}}\exp\Big(\big(D_t^{\mathscr{E}}+dt\,\frac{1}{2t}N^{\Lambda^{\cdot}(\overline{T^*N})}-dt\,\frac{n}{2t}\big)^2\Big)\Big]^{dt}\\
&=\operatorname{Tr_s}\Big[\exp\Big(-C^{\mathscr{E}_+,2}-da\,\frac{1}{2}\big(\overline{\partial}_N^E+t\overline{\partial}_N^{E,*}\big)\\
&\qquad\qquad-d\bar{a}\,\big[\overline{\partial}_N^E+t\overline{\partial}_N^{E,*},\frac{\epsilon}{2}\omega^{\mathscr{E}_+}\big]+dad\bar{a}\,\frac{\epsilon}{2}\omega^{\mathscr{E}_+}\Big)\Big]^{\epsilon dad\bar{a}dt}\\
&\quad-d_M\operatorname{Tr_s}\Big[\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\exp\Big(\big(D_t^{\mathscr{E}}+\frac{1}{2}dt\,N^{\Lambda^{\cdot}(\overline{T^*N})}\big)^2\Big)\Big]^{dt}\\
&\quad+\frac{\partial}{\partial t}\operatorname{Tr_s}\Big[\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\,.
\end{aligned}
\tag{2.3.68}
$$

We multiply (2.3.68) by t and subtract the closed forms. Since dt supercommutes with $N^{\Lambda^{\cdot}(\overline{T^*N})}$ and $D_t^{\mathscr{E}}$, By Proposition 2.3.4, 2.3.5, we can delete the $\frac{n}{2t}$, $dt\,\frac{n}{2t}$ on the left-hand side of (2.3.68). We obtain

$$
\begin{aligned}
&\operatorname{Tr_s}\Big[\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\\
&\quad+\operatorname{Tr_s}\Big[D_t^{\mathscr{E}}\exp\Big(\big(D_t^{\mathscr{E}}+dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big)^2\Big)\Big]^{dt}\\
&=\operatorname{Tr_s}\Big[\exp\Big(-C^{\mathscr{E}_+,2}-da\,\frac{1}{2}\big(\overline{\partial}_N^E+t\overline{\partial}_N^{E,*}\big)\\
&\qquad\qquad-d\bar{a}\,\big[\overline{\partial}_N^E+t\overline{\partial}_N^{E,*},\frac{\epsilon t}{2}\omega^{\mathscr{E}_+}\big]+dad\bar{a}\,\frac{\epsilon t}{2}\omega^{\mathscr{E}_+}\Big)\Big]^{\epsilon dad\bar{a}dt}\\
&\quad+t\frac{\partial}{\partial t}\operatorname{Tr_s}\Big[\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]+\text{closed form}\,.
\end{aligned}
\tag{2.3.69}
$$

We have

$$
\begin{aligned}
&d_M\operatorname{Tr_s}\Big[D_t^{\mathscr{E}}\exp\Big(\big(D_t^{\mathscr{E}}+dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big)^2\Big)\Big]^{dt}\\
&=\operatorname{Tr_s}\Big[\big[C_t^{\mathscr{E}},D_t^{\mathscr{E}}\exp\big(\big(D_t^{\mathscr{E}}+dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big)^2\big)\big]\Big]^{dt}\\
&=-\operatorname{Tr_s}\Big[D_t^{\mathscr{E}}\exp\Big(D_t^{\mathscr{E},2}+\big[C_t^{\mathscr{E}},\big[D_t^{\mathscr{E}},dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big]\big]\Big)\Big]^{dt}\\
&=\operatorname{Tr_s}\Big[D_t^{\mathscr{E}}\exp\Big(D_t^{\mathscr{E},2}+\big[D_t^{\mathscr{E}},\big[C_t^{\mathscr{E}},dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big]\big]\Big)\Big]^{dt}\\
&=\operatorname{Tr_s}\Big[D_t^{\mathscr{E}}\big[D_t^{\mathscr{E}},\exp\big(D_t^{\mathscr{E},2}+\big[C_t^{\mathscr{E}},dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big]\big)\big]\Big]^{dt}\\
&=\operatorname{Tr_s}\Big[2D_t^{\mathscr{E},2}\exp\Big(D_t^{\mathscr{E},2}+\big[C_t^{\mathscr{E}},dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big]\Big)\Big]^{dt}\\
&=\Big(d_M\operatorname{Tr_s}\Big[2D_t^{\mathscr{E},2}\exp\big(D_t^{\mathscr{E},2}+dt\,\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}\big)\Big]\Big)^{dt}\,.
\end{aligned}
\tag{2.3.70}
$$

Thus

$$
\begin{aligned}
&\operatorname{Tr_s}\left[D_t^{\mathscr{E}} \exp\left(\left(D_t^{\mathscr{E}} + dt\,\frac{1}{2}N^{\Lambda^\cdot(\overline{T^*N})}\right)^2\right)\right]^{dt} \\
&= \operatorname{Tr_s}\left[2D_t^{\mathscr{E},2} \exp\left(D_t^{\mathscr{E},2} + dt\,\frac{1}{2}N^{\Lambda^\cdot(\overline{T^*N})}\right)\right]^{dt} + \text{closed form} \\
&= \frac{\partial}{\partial b}\operatorname{Tr_s}\left[N^{\Lambda^\cdot(\overline{T^*N})} \exp\left((1+b)D_t^{\mathscr{E},2}\right)\right]_{b=0} + \text{closed form} \\
&= \frac{\partial}{\partial b}\operatorname{Tr_s}\left[N^{\Lambda^\cdot(\overline{T^*N})} \exp\left((1+b)^{\frac{1}{2}N^{\Lambda^\cdot(T^*M)}} D_{(1+b)t}^{\mathscr{E},2}(1+b)^{-\frac{1}{2}N^{\Lambda^\cdot(T^*M)}}\right)\right]_{b=0} \\
&\quad + \text{closed form} \\
&= \frac{\partial}{\partial b}(1+b)^{\frac{1}{2}N^{\Lambda^\cdot(T^*M)}}\operatorname{Tr_s}\left[N^{\Lambda^\cdot(\overline{T^*N})} \exp\left(D_{(1+b)t}^{\mathscr{E},2}\right)\right]_{b=0} + \text{closed form} \\
&= t\frac{\partial}{\partial t}\operatorname{Tr_s}\left[N^{\Lambda^\cdot(\overline{T^*N})} \exp\left(D_t^{\mathscr{E},2}\right)\right] \\
&\quad + \frac{1}{2}N^{\Lambda^\cdot(T^*M)}\operatorname{Tr_s}\left[N^{\Lambda^\cdot(\overline{T^*N})} \exp\left(D_t^{\mathscr{E},2}\right)\right] + \text{closed form}\,.
\end{aligned}
\tag{2.3.71}
$$

By (2.3.69) and (2.3.71), we get (2.3.66). □

Let r^N be the scalar curvature of (N, g^{TN}). Let R^E, R^{TN} be the curvatures of ∇^E, ∇^{TN} on E, TN over $\mathcal{N}$ so that

$$
R^E = \nabla^{E,2}\,, \qquad R^{TN} = \nabla^{TN,2}. \tag{2.3.72}
$$

Then $\operatorname{Tr}\left[R^{TN}\right]$ is just the curvature of the connection on $\Lambda^n(TN)$ which is induced by ∇^{TN}.

Let $S^{T_{\mathbb{R}}N}$ be the analogue of the tensor S^{TX} in Definition 2.1.9. Since our fibration is flat, it follows from [B86, (1.28)], if $U \in T_{\mathbb{R}}N$ and $V, W \in T^H\mathcal{N}$, then

$$
\left\langle S^{T_{\mathbb{R}}N}(U)\,V, W\right\rangle = \left\langle U, T(V,W)\right\rangle = 0\,. \tag{2.3.73}
$$

Let $\nabla^{\Lambda^\cdot(\overline{T^*N})\otimes E}$ be the connection on $\Lambda^\cdot(\overline{T^*N}) \otimes E$ induced by $\nabla^{\Lambda^\cdot(\overline{T^*N})}$ and E.

We recall that ω is the fiberwise Kähler form, ω^{TN}, ω^E are the variation of metrics on TN, E. We also recall that $c(\cdot)$ is the Clifford action associated to $g^{TN}/2$.

Let $(e_i)_{1\leqslant i\leqslant 2n}$ be an orthonormal basis of $T_{\mathbb{R}}N$, let $(e^i)_{1\leqslant i\leqslant 2n}$ be the corresponding dual basis. Let $(f_\alpha)_{1\leqslant\alpha\leqslant m}$ a basis of TM. We identify the f_α with their horizontal lifts in $T^H\mathcal{N}$. Let $(f^\alpha)_{1\leqslant\alpha\leqslant m}$ be the corresponding dual basis.

To interpret properly the formula that follows, we need to extend the basis e_i to a parallel basis of $T_{\mathbb{R}}N$ near the point x which is considered. Moreover, we may suppose that $\nabla^{T_{\mathbb{R}}N}_{\cdot} e_i = 0$ at the point x.

Proposition 2.3.12. *The following identity holds:*

$$
\begin{aligned}
& - C^{\mathscr{E},2} - da\,\frac{1}{2}(\overline{\partial}_N^E + \overline{\partial}_N^{E,*}) - d\bar{a}\,\big[\overline{\partial}_N^E + \overline{\partial}_N^{E,*}, \frac{\epsilon}{2}\omega^{\mathscr{E}}\big] + dad\bar{a}\,\frac{\epsilon}{2}\omega^{\mathscr{E}} \\
= & \frac{1}{2}\Big(\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*N})\otimes E} + \langle S^{T_{\mathbb{R}}N}(e_i)e_j, f_\alpha\rangle c(e_j) f^\alpha \\
& \qquad - da\,\frac{1}{2}c(e_i) - d\bar{a}\epsilon\,\frac{\sqrt{-1}}{2}(d_M\omega)(e_i,e_j)c(e_j)\Big)^2 \\
& - d\bar{a}\epsilon\,\big[\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*N})\otimes E}, \frac{1}{2}\omega^E - \frac{1}{8}(d_M\omega^{TN})(e_j, Je_j)\big]c(e_i) \\
& + dad\bar{a}\epsilon\Big(\frac{1}{2}\omega^E - \frac{1}{8}(d_M\omega)(e_j,Je_j)\Big) \\
& - \frac{1}{2}\Big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\Big)(e_i,e_j)c(e_i)c(e_j) - \Big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\Big)(e_i,f_\alpha)c(e_i)f^\alpha \\
& - \frac{1}{2}\Big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\Big)(f_\alpha,f_\beta)f^\alpha f^\beta - \frac{1}{8}r^N\ .
\end{aligned}
\tag{2.3.74}
$$

Proof. Applying [B86, Theorem 3.5] with $t = 1/\sqrt{2}$ and (2.3.73), we have

$$
\begin{aligned}
& - C^{\mathscr{E},2} \\
= & \frac{1}{2}\Big(\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*N})\otimes E} + \langle S^{T_{\mathbb{R}}N}(e_i)e_j, f_\alpha\rangle c(e_j) f^\alpha\Big)^2 \\
& - \frac{1}{2}\big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\big)(e_i,e_j)c(e_i)c(e_j) - \big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\big)(e_i,f_\alpha)c(e_i)f^\alpha \\
& - \frac{1}{2}\big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\big)(f_\alpha,f_\beta)f^\alpha f^\beta - \frac{1}{8}r^N\ .
\end{aligned}
\tag{2.3.75}
$$

Taking the degree 0 part of (2.3.75), we get

$$
\begin{aligned}
& - \big(\overline{\partial}_N^E + \overline{\partial}_N^{E,*}\big)^2 \\
= & \frac{1}{2}\Big(\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*N})\otimes E}\Big)^2 - \frac{1}{2}\big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\big)(e_i,e_j)c(e_i)c(e_j) - \frac{1}{8}r^N\ .
\end{aligned}
\tag{2.3.76}
$$

By [BGS88c, Proposition 1.19] and by (2.3.26), we get

$$
\omega^{\mathscr{E}} = -\frac{\sqrt{-1}}{2}(d_M\omega)(e_i,e_j)c(e_i)c(e_j) - \frac{1}{4}(d_M\omega)(e_i,Je_i) + \omega^E\ .
\tag{2.3.77}
$$

By $d_N\omega = 0$ and $[d_N, d_M] = 0$, we have $d_Nd_M\omega = 0$. Therefore

$$
\begin{aligned}
& \big[\overline{\partial}_N^E + \overline{\partial}_N^{E,*}, -\frac{\epsilon\sqrt{-1}}{4}(d_M\omega)(e_i,e_j)c(e_i)c(e_j)\big] \\
= & \frac{\epsilon\sqrt{-1}}{4}\big[c(e_k)\nabla_{e_k}^{\Lambda^\cdot(\overline{T^*N})\otimes E}, (d_M\omega)(e_i,e_j)c(e_i)c(e_j)\big] \\
= & \frac{\epsilon\sqrt{-1}}{4}\Big(\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*N})\otimes E}(d_M\omega)(e_i,e_j)c(e_j) + (d_M\omega)(e_i,e_j)c(e_j)\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*N})\otimes E}\Big)\ .
\end{aligned}
\tag{2.3.78}
$$

By (2.3.76), (2.3.77) and (2.3.78), we get

$$
\begin{aligned}
&- \big(\overline{\partial}_N^E + \overline{\partial}_N^{E,*}\big)^2 - da\,\frac{1}{2}\big(\overline{\partial}_N^E + \overline{\partial}_N^{E,*}\big) - d\bar a\,\big[\overline{\partial}_N^E + \overline{\partial}_N^{E,*}, \frac{\epsilon}{2}\omega^{\mathscr{E}}\big] + dad\bar a\,\frac{\epsilon}{2}\omega^{\mathscr{E}} \\
&= \frac{1}{2}\Big(\nabla_{e_i}^{\Lambda^\cdot \overline{T^*N}\otimes E} - da\,\frac{1}{2}c(e_i) - d\bar a\epsilon\,\frac{\sqrt{-1}}{2}(d_M\omega)(e_i,e_j)c(e_j)\Big)^2 \\
&\quad - d\bar a\epsilon\,\big[\nabla_{e_i}^{\Lambda^\cdot(\overline{T^*N})\otimes E}, \frac{1}{2}\omega^E - \frac{1}{8}(d_M\omega^{TN})(e_j, Je_j)\big]c(e_i) \qquad (2.3.79)\\
&\quad + dad\bar a\epsilon\,\big(\frac{1}{2}\omega^E - \frac{1}{8}(d_M\omega)(e_j,Je_j)\big) \\
&\quad - \frac{1}{2}\big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TN}]\big)(e_i,e_j)c(e_i)c(e_j) - \frac{1}{8}r^N\ .
\end{aligned}
$$

Comparing (2.3.75), (2.3.76), (2.3.79) with (2.3.74), it only remains to show that

$$
\begin{aligned}
&\sum_{i\neq j}\langle S^{T_{\mathbb{R}}N}(e_i)e_j, f_\alpha\rangle f^\alpha c(e_i)c(e_j) = 0\ , \\
&\sum_i\sum_{j\neq k}(d_M\omega)(e_i,e_j)\langle S^{T_{\mathbb{R}}N}(e_i)e_k, f_\alpha\rangle f^\alpha c(e_j)c(e_k) = 0\ .
\end{aligned} \qquad (2.3.80)
$$

The first identity in (2.3.80) follows from the fact that (cf. [B86, §1(c)]) if $U, V\in T\mathcal{N}$ $S^{T_{\mathbb{R}}N}(U)V - S^{T_{\mathbb{R}}N}(V)U \in T_{\mathbb{R}}N$, then

$$
\langle S^{T_{\mathbb{R}}N}(e_i)e_j, f_\alpha\rangle = \langle S^{T_{\mathbb{R}}N}(e_j)e_i, f_\alpha\rangle\ . \qquad (2.3.81)
$$

By (2.3.81), we get the first identity in (2.3.80).

Now, we prove the second identity in (2.3.80). By [B97, (1.5)], we have

$$
\langle S^{T_{\mathbb{R}}N}(e_i)e_k, f_\alpha\rangle = -\frac{1}{2}\Big\langle \big(g^{T_{\mathbb{R}}N}\big)^{-1}\nabla_{f_\alpha} g^{T_{\mathbb{R}}N}(e_i)\,,\, e_k\Big\rangle = -\frac{1}{2}\left(\nabla_{f_\alpha}\omega\right)(e_i, Je_k)\ . \qquad (2.3.82)
$$

Therefore the second identity in (2.3.80) is equivalent to the following one :

$$
\sum_i\sum_{j\neq k}(\nabla_{f_\alpha}\omega)(e_i,e_j)(\nabla_{f_\beta}\omega)(e_i,Je_k)f^\alpha f^\beta c(e_j)c(e_k) = 0\ . \qquad (2.3.83)
$$

Since $(Je_i)_{1\leqslant i\leqslant n}$ is also an orthogonal basis of $T_{\mathbb{R}}N$, using the fact that ω and $d_M\omega$ are J-invariant, we get

$$
\begin{aligned}
&\sum_i\sum_{j\neq k}\left(\nabla_{f_\alpha}\omega\right)(e_i,e_j)\left(\nabla_{f_\beta}\omega\right)(e_i,Je_k)f^\alpha f^\beta c(e_j)c(e_k) \\
&= \frac{1}{2}\sum_i\sum_{j\neq k}\left(\nabla_{f_\alpha}\omega\right)(e_i,e_j)\left(\nabla_{f_\beta}\omega\right)(e_i,Je_k)f^\alpha f^\beta c(e_j)c(e_k) \\
&\quad + \frac{1}{2}\sum_i\sum_{j\neq k}\left(\nabla_{f_\alpha}\omega\right)(Je_i,e_j)\left(\nabla_{f_\beta}\omega\right)(Je_i,Je_k)f^\alpha f^\beta c(e_j)c(e_k) \qquad (2.3.84)\\
&= \frac{1}{2}\sum_i\sum_{j\neq k}\left(\nabla_{f_\alpha}\omega\right)(e_i,e_j)\left(\nabla_{f_\beta}\omega\right)(e_i,Je_k)f^\alpha f^\beta c(e_j)c(e_k) \\
&\quad - \frac{1}{2}\sum_i\sum_{j\neq k}\left(\nabla_{f_\alpha}\omega\right)(e_i,Je_j)\left(\nabla_{f_\beta}\omega\right)(e_i,e_k)f^\alpha f^\beta c(e_j)c(e_k)\ .
\end{aligned}
$$

By exchanging the roles of j,k and α,β, we obtain

$$\begin{aligned}(2.3.85)\qquad &\sum_i \sum_{j\neq k} \left(\nabla_{f_\alpha}\omega\right)(e_i, Je_j)\left(\nabla_{f_\beta}\omega\right)(e_i,e_k) f^\alpha f^\beta c(e_j)c(e_k)\\ &= \sum_i \sum_{j\neq k} \left(\nabla_{f_\beta}\omega\right)(e_i, Je_k)\left(\nabla_{f_\alpha}\omega\right)(e_i,e_j) f^\beta f^\alpha c(e_k)c(e_j)\\ &= \sum_i \sum_{j\neq k} \left(\nabla_{f_\alpha}\omega\right)(e_i, e_j)\left(\nabla_{f_\beta}\omega\right)(e_i,Je_k) f^\alpha f^\beta c(e_j)c(e_k)\,.\end{aligned}$$

By (2.3.84), (2.3.85), we get (2.3.83). □

Proof of Theorem 2.3.7. The proof of (2.3.56) follows the same argument as [BL95, Theorem 3.16].

We turn to prove the first formula in (2.3.57).

By Lemma 2.3.10, it is sufficient to establish the asymptotic expansion of the following two terms as $t \to 0$:

$$\begin{aligned}(2.3.86)\qquad &\mathrm{Tr_s}\Big[\exp\big(-C_t^{\mathscr{E},2} - da\,\frac{1}{2}(\overline{\partial}_N^E + t\overline{\partial}_N^{E,*})\\ &\qquad - d\bar{a}\,\big[\overline{\partial}_N^E + t\overline{\partial}_N^{E,*}, \frac{\epsilon}{2}\omega^{\mathscr{E}}\big] + dad\bar{a}\,\frac{\epsilon}{2}\omega^{\mathscr{E}}\big)\Big]^{\epsilon dad\bar{a}}\,,\\ &d_M \,\mathrm{Tr_s}\Big[\frac{1}{2}N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\,.\end{aligned}$$

As $t \to 0$, we claim that we can use equation (2.3.74) exactly as in Bismut-Köhler [BK92, Theorem 3.22]. The main difference is that in [BK92], the space of variations of the metrics is 1-dimensional, while here it is the full basis M. By proceeding as in this reference, we get

$$\begin{aligned}(2.3.87)\qquad &\sqrt{2\pi i}\varphi\,\mathrm{Tr_s}\Big[\exp\big(-C_t^{\mathscr{E},2} - da\,\frac{1}{2}(\overline{\partial}_N^E + t\overline{\partial}_N^{E,*})\\ &\qquad - d\bar{a}\,\big[\overline{\partial}_N^E + t\overline{\partial}_N^{E,*}, \frac{\epsilon}{2}\omega^{\mathscr{E}}\big] + dad\bar{a}\,\frac{\epsilon}{2}\omega^{\mathscr{E}}\big)\Big]^{\epsilon dad\bar{a}}\\ &= q_*\Big[\widetilde{\mathrm{Td}}(TN, g^{TN}) * \widetilde{\mathrm{ch}}(E, g^E)\Big] + \mathscr{O}(t)\,.\end{aligned}$$

This gives the asymptotic expansion of the first term in (2.3.86).

We will study the second term in (2.3.86). As $t \to 0$, by the local families index theorem technique [B86], we get

$$\begin{aligned}(2.3.88)\qquad &\varphi\,\mathrm{Tr_s}\Big[tN^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\\ &= q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\Big] + \mathscr{O}(\sqrt{t})\,.\end{aligned}$$

Furthermore, by [BGS88b, Theorems 2.11 and 2.16], the asymptotic expansion of $\mathrm{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]$ is a Laurent series on t. By (2.3.88), we get

$$(2.3.89)\qquad \varphi\,\mathrm{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big] = C_{-1}t^{-1} + C_0 + \mathscr{O}(t)\,,$$

with

$$(2.3.90)\qquad C_{-1} = q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\Big]\,.$$

Let $C_{-1}^{(p)}$ (resp. $C_0^{(p)}$) be the component of degree p of C_{-1} (resp. C_0). By Remark 2.3.8, for $p > 0$, $C_{-1}^{(p)} = 0$. Then

$$\begin{aligned}&\big(1 + N^{\Lambda^\cdot(T^*M)} + t\frac{\partial}{\partial t}\big)\,\mathrm{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\\ &= \sum_p \Big((p+1)C_0^{(p)}\Big) + \mathscr{O}(t)\ .\end{aligned} \tag{2.3.91}$$

Applying (2.3.87) with $\mathscr{E}$ replaced by $\mathscr{E}_+$ (see the proof of Proposition 2.3.6) and taking the dt component, we get

$$\begin{aligned}&\varphi\,\mathrm{Tr_s}\Big[\exp\big(-C^{\mathscr{E}_+,2} - da\,\frac{1}{2}(\overline{\partial}_N^E + t\overline{\partial}_N^{E,*})\\ &\qquad - d\bar{a}\,\big[\overline{\partial}_N^E + t\overline{\partial}_N^{E,*}, \frac{\epsilon t}{2}\omega^{\mathscr{E}_+}\big] + dad\bar{a}\,\frac{\epsilon t}{2}\omega^{\mathscr{E}_+}\big)\Big]^{\epsilon dad\bar{a}dt}\\ &= -\frac{1}{2}q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big] + \mathscr{O}(t)\ .\end{aligned} \tag{2.3.92}$$

By Theorem 2.2.5, Lemma 2.3.11 and (2.3.92), we have

$$\big(1 + N^{\Lambda^\cdot(T^*M)} + t\frac{\partial}{\partial t}\big)\,\mathrm{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big] = \text{ closed form } + \mathscr{O}(t)\ . \tag{2.3.93}$$

By (2.3.91) and (2.3.93), we have

$$d_M C_0 = 0\ . \tag{2.3.94}$$

By (2.3.89), (2.3.90), (2.3.94), as $t \to 0$, we have

$$\begin{aligned}&\sqrt{2\pi i}\varphi\, d_M\,\mathrm{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\\ &= d_M\varphi\,\mathrm{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(D_t^{\mathscr{E},2}\big)\Big]\\ &= \frac{1}{t}d_M q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big] + \mathscr{O}(\sqrt{t})\ .\end{aligned} \tag{2.3.95}$$

The first formula in (2.3.57) follows from Lemma 2.3.10, (2.3.87) and (2.3.95).

The second formula in (2.3.57) may be proved as a consequence of the first one by applying the same technique as the proof of Proposition 2.3.6. □

2.3.7. *Higher analytic torsion forms.*

We choose $g_1, g_2 \in \mathscr{C}^\infty(\mathbb{R}_+, \mathbb{R})$ satisfying

$$g_1(t) = 1 + \mathscr{O}\big(t\big)\ ,\quad g_2(t) = 1 + \mathscr{O}\big(t^2\big)\ ,\quad \text{as } t \to 0\ , \tag{2.3.96}$$

$$g_1(t) = \mathscr{O}\big(e^{-t}\big)\ ,\quad g_2(t) = \mathscr{O}\big(e^{-t}\big)\ ,\quad \text{as } t \to +\infty\ , \tag{2.3.97}$$

and

$$\begin{aligned}&\int_0^1 \frac{g_1(t)-1}{t}dt + \int_1^{+\infty}\frac{g_1(t)}{t} = \Gamma'(1) - 2\ ,\\ &\int_0^1 \frac{g_2(t)-1}{t^2}dt + \int_1^{+\infty}\frac{g_2(t)}{t^2} = 1\ .\end{aligned} \tag{2.3.98}$$

Using Mellin tranformation, (2.3.98) is reformulated as follows

$$\begin{aligned}
&\Big(\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}g_1(t)dt\Big)_{s=0} = -2\;,\\
&\Big(\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-2}g_2(t)dt\Big)_{s=0} = 0\;.
\end{aligned}\tag{2.3.99}$$

Definition 2.3.13. The analytic torsion forms $\mathscr{T}(g^{TN},g^E)\in\Omega^{\mathrm{even}}(M)$ are defined by

$$\begin{aligned}
\mathscr{T}(g^{TN},g^E) = -\int_0^{+\infty}\Big\{\beta_t &+ \frac{g_1(t)-1}{2}\chi'(N,E) - \frac{g_1(t)}{2}n\chi(N,E)\\
&+\frac{g_1(t)}{2}q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]\\
&+\frac{g_2(t)}{2t}q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]\Big\}\frac{dt}{t}\;.
\end{aligned}\tag{2.3.100}$$

By Theorem 2.3.7, $\mathscr{T}(g^{TN},g^E)$ is well-defined. Moreover, $\mathscr{T}(g^{TN},g^E)$ is independent of g_1 and g_2.

Proposition 2.3.14. *We have*

$$\begin{aligned}
d_M\mathscr{T}(g^{TN},g^E) = q_*&\Big[\widetilde{\mathrm{Td}}(TN,g^{TN})*\widetilde{\mathrm{ch}}(E,g^E)\Big]\\
&- f(H^\cdot(N,E),\nabla^{H^\cdot(N,E)},g^{H^\cdot(N,E)})\;.
\end{aligned}\tag{2.3.101}$$

Proof. By Theorem 2.2.5, $q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]$ is a constant function on M. Then, by Proposition 2.3.6, we get

$$\begin{aligned}
&d_M\mathscr{T}(g^{TN},g^E)\\
&= -\int_0^{+\infty}\Big\{d_M\beta_t + \frac{g_2(t)}{2t}d_Mq_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]\Big\}\frac{dt}{t}\\
&= -\int_0^{+\infty}\Big\{\frac{\partial}{\partial t}\alpha_t + \frac{g_2(t)}{2t^2}d_Mq_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]\Big\}dt\;.
\end{aligned}\tag{2.3.102}$$

By Theorem 2.3.7, the second identity in (2.3.98) and (2.3.102), we get (2.3.101). □

Proceeding in the same way as [BL95, Theorem 3.16], we get

$$\mathrm{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(-tD_N^{E,2}\big)\Big] = \chi'(N,E)+\mathscr{O}\big(t^{-1}\big)\;,\quad \text{as } t\to+\infty\;.\tag{2.3.103}$$

For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n$, we define

$$\begin{aligned}
&\theta(s)\\
&= -\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\Big[\mathrm{Tr_s}\big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(-tD_N^{E,2}\big)\big]-\chi'(N,E)\Big]dt\;.
\end{aligned}\tag{2.3.104}$$

By [See67], $\theta(s)$ admits a meromorphic extension to the whole complex plane, which is regular at $0\in\mathbb{C}$.

Let $\mathscr{T}^{[0]}(g^{TN},g^E)$ be the component of $\mathscr{T}(g^{TN},g^E)$ of degree zero.

Proposition 2.3.15. *We have*

$$\mathscr{T}^{[0]}(g^{TN},g^E) = \frac{1}{2}\theta'(0)\;.\tag{2.3.105}$$

Proof. By (2.3.35) and (2.3.45), we get

$$\begin{aligned}\beta_t^{[0]} &= \operatorname{Tr_s}\Big[\frac{N^{\Lambda^\cdot(\overline{T^*N})}}{2}\big(1-2tD_N^{E,2}\big)\exp\big(-tD_N^{E,2}\big)\Big]\\ &=\frac{1}{2}\Big(1+2t\frac{\partial}{\partial t}\Big)\operatorname{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(-tD_N^{E,2}\big)\Big]\,.\end{aligned} \tag{2.3.106}$$

By (2.3.89), as $t\to 0$, we have

$$\operatorname{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(-tD_N^{E,2}\big)\Big]=a_{-1}t^{-1}+a_0+\mathscr{O}\big(\sqrt{t}\big)\,. \tag{2.3.107}$$

By (2.3.57), (2.3.106), (2.3.107), we get

$$a_0=-q_*\big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\big]+n\chi(N,E)\,. \tag{2.3.108}$$

By (2.3.104), (2.3.107), (2.3.108), we get

$$\theta(0)=q_*\big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\big]-n\chi(N,E)+\chi'(N,E)\,. \tag{2.3.109}$$

By Definition 2.3.13, (2.3.99), (2.3.104), (2.3.106), we have

$$\begin{aligned}&\mathscr{T}^{[0]}(g^{TN},g^E)\\ &=-\int_0^{+\infty}\Big\{\frac{1}{2}\Big(1+2t\frac{\partial}{\partial t}\Big)\operatorname{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(-tD_N^{E,2}\big)\Big]-\frac{1}{2}\chi'(N,E)\\ &\qquad+\frac{g_1(t)}{2}\Big(q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]-n\chi(N,E)+\chi'(N,E)\Big)\\ &\qquad+\frac{g_2(t)}{2t}q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]\Big\}\frac{dt}{t}\\ &=-\frac{1}{2}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\Big(1+2t\frac{\partial}{\partial t}\Big)\Big\{\operatorname{Tr_s}\Big[N^{\Lambda^\cdot(\overline{T^*N})}\exp\big(-tD_N^{E,2}\big)\Big]\\ &\qquad\qquad-\chi'(N,E)\Big\}dt\\ &\quad-\frac{1}{2}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}g_1(t)dt\,\Big(q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]\\ &\qquad\qquad-n\chi(N,E)+\chi'(N,E)\Big)\\ &\quad-\frac{1}{2}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-2}g_2(t)dt\,q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]\\ &=\frac{d}{ds}\Big|_{s=0}\frac{1-2s}{2}\theta(s)+q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]-n\chi(N,E)+\chi'(N,E)\\ &=\frac{1}{2}\theta'(0)-\theta(0)+q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E,\nabla^E)\Big]-\chi(N,E)+\chi'(N,E)\,.\end{aligned} \tag{2.3.110}$$

By (2.3.109) and (2.3.110), we obtain (2.3.105). □

2.4. The analytic torsion forms of a bicomplex.

In this subsection, we define the analytic torsion forms over S of a bicomplex where the differential is $d_M+\overline{\partial}_N$. A fiberwise positive line bundle L plays a critical role in our constructions. To define the analytic torsion forms, we use a nondegeneracy assumption made in Bismut-Ma-Zhang [BMaZ15] that guarantees that the de Rham cohomogy of the fibre X with coefficients in the considered flat vector bundle on M vanishes identically.

These analytic torsion forms can be thought as the analytic torsion forms of Bismut-Lott [BL95] with coefficients in an infinite dimensional flat vector bundle.

Also we use an adiabatic limit technique to express these analytic torsion forms on S in terms of the analytic torsion forms of Bismut-Lott [BL95] of the direct image, and of the analytic torsion forms that we obtained in §2.3.7.

This section is organized as follows.

In §2.4.1, we construct the spectral sequence associated with the fibration $q : \mathcal{N} \to M$.

In §2.4.2, we construct a flat superconnection, which is a version of the construction of Bismut-Lott [BL95], where the considered flat vector bundle is itself of infinite dimension.

In §2.4.3, we equip TN, TX, E with metrics.

In §2.4.4, we construct a Levi-Civita superconnection, which is again an extension of constructions of Bismut-Lott [BL95].

In §2.4.5, we introduce the Hermitian line bundle (L, g^L) on $\mathcal{N}$, on which we make the nondegeneracy assumption of Bismut-Ma-Zhang [BMaZ15].

In §2.4.6, we recall some results of Bismut-Lott [BL95].

Finally, in §2.4.7, we state our main results. Their proofs are delayed to section 2.5.

2.4.1. *A filtered complex and its spectral sequence.*

We recall that the fibration $q : \mathcal{N} \to M$ and the (infinite dimensional) flat vector bundle $(\mathscr{E}^{\cdot}, \nabla^{\mathscr{E}})$ over M are defined in §2.3.1.

For any $p, q \geqslant 0$, set

$$\Omega^{p,q}(\mathcal{N}, E) = \mathscr{C}^{\infty}\left(\mathcal{N}, \Lambda^p(T^*M) \otimes \Lambda^q(\overline{T^*N}) \otimes E\right) = \Omega^p\left(M, \mathscr{E}^q\right) . \tag{2.4.1}$$

Then $\left(\Omega^{\cdot,\cdot}(\mathcal{N}, E), \overline{\partial}^E_N, \nabla^{\mathscr{E}}\right)$ is a bicomplex.

For any $k \geqslant 0$, set

$$\Omega^k(\mathcal{N}, E) = \bigoplus_{p+q=k} \Omega^{p,q}(\mathcal{N}, E) . \tag{2.4.2}$$

Set

$$D'' = \overline{\partial}^E_N + \nabla^{\mathscr{E}} . \tag{2.4.3}$$

Then, $(\Omega^{\cdot}(\mathcal{N}, E), D'')$ is a simple complex. Let $H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E)$ be the cohomology of $(\Omega^{\cdot}(\mathcal{N}, E), D'')$. We remark that $(\Omega^{\cdot}(\mathcal{N}, E), D'')$ is an elliptic complex, as a consequence, $H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E)$ is finite dimensional if M is compact.

Let

$$\begin{aligned} \Omega^{\cdot}(\mathcal{N}, E) = F^0\Omega^{\cdot}(\mathcal{N}, E) \supseteq F^1\Omega^{\cdot}(\mathcal{N}, E) \supseteq \cdots \\ \supseteq F^{\dim M+1}\Omega^{\cdot}(\mathcal{N}, E) = \{0\} . \end{aligned} \tag{2.4.4}$$

be the a filtration of $\Omega^{\cdot}(\mathcal{N}, E)$, given by

$$F^p\Omega^k(\mathcal{N}, E) = \bigoplus_{\substack{p' \geqslant p \\ p'+q'=k}} \Omega^{p',q'}(\mathcal{N}, E) . \tag{2.4.5}$$

Then $(\Omega^{\cdot}(\mathcal{N}, E), D'', F^{\cdot})$ is a filtered complex. Let

$$\begin{aligned} H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E) = F^0H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E) \supseteq F^1H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E) \supseteq \cdots \\ \supseteq F^{\dim M+1}H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E) = \{0\} . \end{aligned} \tag{2.4.6}$$

be the induced filtration on $H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E)$.

For any $p \geqslant 0$, set

$$\mathrm{Gr}^p H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E) = \frac{F^p H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E)}{F^{p+1} H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E)} \ . \tag{2.4.7}$$

Let $\left(E_r^{p,q}, d_r : E_r^{p,q} \to E_r^{p-r,q+r+1}\right)_{r \geqslant 0}$ be the spectral sequence associated to the filtration $F^{\cdot}$ on the complex $(\Omega^{\cdot}(\mathcal{N}, E), D'')$.

We have

$$\begin{aligned} E_0^{p,q} &= \Omega^p\left(M, \Omega^q(N, E)\right) \ , \\ (E_0, d_0) &= \left(\Omega^{\cdot}\left(M, \Omega^{\cdot}(N, E)\right), \overline{\partial}_N^E\right) \ . \end{aligned} \tag{2.4.8}$$

Recall that $H^{\cdot}(N, E)$ is the fiberwise Dolbeault cohomology of the fibration $q : \mathcal{N} \to M$ with coefficient in E, which is also a flat vector bundle over M with flat connection $\nabla^{H^{\cdot}(N,E)}$. The de Rham operator acting on $\Omega^{\cdot}\left(M, H^{\cdot}(N, E)\right)$ is also denoted by $\nabla^{H^{\cdot}(N,E)}$. We have

$$\begin{aligned} E_1^{p,q} &= \Omega^p\left(M, H^q(N, E)\right) \ , \\ (E_1, d_1) &= \left(\Omega^{\cdot}\left(M, H^{\cdot}(N, E)\right), \nabla^{H^{\cdot}(N,E)}\right) \ . \end{aligned} \tag{2.4.9}$$

For any $q \geqslant 0$, let $H^{\cdot}(M, H^q(N, E))$ be the cohomology of M with coefficients in the flat vector bundle $H^q(N, E)$. We have

$$E_2^{p,q} = H^p(M, H^q(N, E)) \ . \tag{2.4.10}$$

For $r > \min\left\{\dim M, \dim N\right\}$, we have

$$\begin{aligned} E_r^{p,q} &= \mathrm{Gr}^p H^{p+q}_{\mathrm{tot}}(\mathcal{N}, E) \ , \\ (E_r, d_r) &= (\mathrm{Gr}^{\cdot} H^{\cdot}_{\mathrm{tot}}(\mathcal{N}, E), 0) \ . \end{aligned} \tag{2.4.11}$$

Remark 2.4.1. If M is compact, $E_2^{p,q} = H^p(M, H^q(N, E))$ is finite dimensional. This provides another proof that $H^q_{\mathrm{tot}}(\mathcal{N}, E)$ is finite dimensional for M compact.

Remark 2.4.2. If $H^q(N, E) = 0$ for $q > 0$, the spectral sequence $(E_r, d_r)_{r \geqslant 0}$ degenerates at $r = 2$. Then, for $r > \min\left\{\dim M, \dim N\right\}$, we have

$$H^p(M, H^0(N, E)) = E_2^{p,\cdot} = E_r^{p,\cdot} = H^p_{\mathrm{tot}}(\mathcal{N}, E) \ . \tag{2.4.12}$$

2.4.2. *A double fibration and a flat superconnection.*

Let

$$\pi : \ M \to S \tag{2.4.13}$$

be a real smooth fibration with compact fibers. For $s \in S$, set $X_s = \pi^{-1}(s)$.

The composition map

$$r = \pi \circ q : \ \mathcal{N} \to S \tag{2.4.14}$$

is again a fibration. For $s \in S$, set $Y_s = r^{-1}(s)$.

Let $q_s : Y_s \to X_s$ be the restriction of q.

The objects concerned above fit into the following commutative diagram.

$$\begin{tikzcd} \mathcal{N} \arrow[r, "q"] & M \arrow[r, "\pi"] & S \\ Y_s \arrow[u, hook] \arrow[r, "q_s"] & X_s \arrow[u, hook] \arrow[r] & \{s\} \arrow[u, hook] \end{tikzcd} \tag{2.4.15}$$

In the sequel, we will systematically omit the subscript s.

We recall that the fibration $q : \mathcal{N} \to M$ is equipped with a flat connection, defined by (2.2.7). By restricting (2.2.7) to Y, we get

$$TY = T_{\mathbb{R}}N \oplus T^H Y \ , \quad T^H Y \simeq q^* TX \ . \tag{2.4.16}$$

We have the following identification induced by (2.4.16)

$$\Omega^{p,q}(Y,E) = \Omega^p(X, \mathscr{E}^q) \ , \tag{2.4.17}$$

where $\Omega^{p,q}(\cdot, E)$ is defined by (2.4.1).

We equip the fibration $\pi : M \to S$ with a connection, i.e., with a smooth a splitting

$$TM = TX \oplus T^H M \ , \quad T^H M \simeq \pi^* TS \ . \tag{2.4.18}$$

For $U \in TS$, we denote by $U^H \in T^H M$ the lifting of U in $T^H M$, i.e., the unique vector satisfying $\pi_* U^H = U$.

Set

$$\mathscr{F}^{p,q} = \Omega^{p,q}(Y,E) \ , \quad \mathscr{F} = \bigoplus_{p,q} \mathscr{F}^{p,q} \ , \tag{2.4.19}$$

which are infinite dimensional vector bundles over S. We have the following identification induced by (2.4.18)

$$\Omega^{p,q}(\mathcal{N},E) = \bigoplus_{p'+p''=p} \Omega^{p'}(S, \mathscr{F}^{p'',q}) \ . \tag{2.4.20}$$

The identifications introduced are summarized as follows.

$$\begin{aligned} &\Omega^q(N,E) = \mathscr{E}^q \ , \\ &\Omega^{p,q}(Y,E) = \Omega^p(X,\mathscr{E}^q) = \mathscr{F}^{p,q} \ , \\ &\Omega^{p,q}(\mathcal{N},E) = \Omega^p(M,\mathscr{E}^q) = \bigoplus_{p'+p''=p} \Omega^{p'}(S,\mathscr{F}^{p'',q}) \ . \end{aligned} \tag{2.4.21}$$

We recall that operator $A^{\mathscr{E}}$ acting on $\Omega^{\cdot}(M,\mathscr{E})$ is defined by (2.3.3). Passing though the identification $\Omega^{\cdot}(M,\mathscr{E}) = \Omega^{\cdot}(S,\mathscr{F})$ (cf. (2.4.21)), $A^{\mathscr{E}}$ defines an action on $\Omega^{\cdot}(S,\mathscr{F})$, denoted by $A^{\mathscr{F}}$.

We recall that $\overline{\partial}^E_N$ is the Dolbeault operator acting on $\mathscr{E} = \mathscr{C}^\infty(N, \Lambda^{\cdot}(\overline{T^*N}) \otimes E)$, defined in §2.2.2. We recall that $\nabla^{\mathscr{F}}$ is the flat connection on $\mathscr{E}$ over M, defined in §2.3.1. Passing though the identifications (2.4.21), both $\overline{\partial}^E_N$ and $\nabla^{\mathscr{E}}$ act on $\Omega^{\cdot}(S,\mathscr{F})$. Then, by (2.3.3), we have

$$A^{\mathscr{F}} = \overline{\partial}^E_N + \nabla^{\mathscr{E}} \ . \tag{2.4.22}$$

Let d_X be the de Rham operator acting on $\Omega^{\cdot}(X)$. Its extension to $\Omega^{\cdot}(X,\mathscr{E})$ is denote by $d^{\mathscr{E}}_X$.

For V a vector field on M, let L_V be the Lie derivative acting on $\Omega^{\cdot}(M)$. Its extension to $\Omega^{\cdot}(M,\mathscr{E})$ is still denoted by L_V. For $U \in TS$ and $\xi \in \mathscr{C}^\infty(S,\mathscr{F}) = \mathscr{C}^\infty(S, \Omega^{\cdot}(X,\mathscr{E})) \subseteq \Omega^{\cdot}(M,\mathscr{E})$, set

$$\nabla^{\mathscr{F}}_U \xi = L_{U^H} \xi \ . \tag{2.4.23}$$

Then $\nabla^{\mathscr{F}}$ is a connection on the infinite dimensional vector bundle $\mathscr{F}$ over S.

Let $T \in \Omega^2(S, \mathscr{C}^\infty(X,TX))$ be the curvature of the fibration, defined in §2.1.4. Then i_T acts on $\Omega^{\cdot}(S,\Omega^{\cdot}(X))$. This action extends to $\Omega^{\cdot}(S,\Omega^{\cdot}(X,\mathscr{E})) = \Omega^{\cdot}(S,\mathscr{F})$.

Passing though the identifications (2.4.21), all the operators $d_X^{\mathscr{E}}$, $\nabla^{\mathscr{F}}$ and i_T act on $\Omega^{\cdot}(S,\mathscr{F})$. By [BL95, Proposition 3.4], we have

$$\nabla^{\mathscr{E}} = d_X^{\mathscr{E}} + \nabla^{\mathscr{F}} + i_T \ . \tag{2.4.24}$$

Then

$$A^{\mathscr{F}} = \overline{\partial}_N^E + d_X^{\mathscr{E}} + \nabla^{\mathscr{F}} + i_T \ . \tag{2.4.25}$$

For $k\in\mathbb{N}$, let $A^{\mathscr{F},[k]} : \Omega^{\cdot}(S,\mathscr{F}) \to \Omega^{\cdot+k}(S,\mathscr{F})$ be the degree k component of $A^{\mathscr{F}}$, then

$$A^{\mathscr{F}} = A^{\mathscr{F},[0]} + A^{\mathscr{F},[1]} + A^{\mathscr{F},[2]} \ , \tag{2.4.26}$$

with

$$A^{\mathscr{F},[0]} = \overline{\partial}_N^E + d_X^{\mathscr{E}} \ , \quad A^{\mathscr{F},[1]} = \nabla^{\mathscr{F}} \ , \quad A^{\mathscr{F},[2]} = i_T \ . \tag{2.4.27}$$

Since $A^{\mathscr{F},[1]}$ is a connection, $A^{\mathscr{F}}$ is a superconnection on $\mathscr{F}$ over S. Moreover, by (2.3.5), we have

$$A^{\mathscr{F},2} = 0 \ , \tag{2.4.28}$$

i.e., $A^{\mathscr{F}}$ is a flat superconnection.

2.4.3. *Metrics on TN, TX and Clifford actions.*
Set

$$\mathscr{S} = \Lambda^{\cdot}(T^*X) \otimes \Lambda^{\cdot}(\overline{T^*N}) \ . \tag{2.4.29}$$

Still, let g^{TN} be a fiberwise Kähler metric on TN. Let g^{TX} be a Riemannian metric on TX. Then $\mathscr{S}$ is equipped with the actions of $C(T_{\mathbb{R}}N, \frac{1}{2}g^{T_{\mathbb{R}}N})$, $C(TX,g^{TX})$, $\widehat{C}(TX,g^{TX})$, defined in §2.1.2.

Let $g^{\mathscr{S}}$ be the metric on $\mathscr{S}$ induced by g^{TX} and g^{TN}.

We recall that the connection $\nabla^{T_{\mathbb{R}}N}$ on $T_{\mathbb{R}}N$ is defined in §2.3.2. Let ∇^{TX} be the Levi-Civita connection on TX with respect to g^{TX}. Let ∇^{TY} be the connection on $TY = T_{\mathbb{R}}N \oplus TX$ induced by $\nabla^{T_{\mathbb{R}}N}$ and ∇^{TX}. Let $\nabla^{\mathscr{S}}$ be the connection on $\mathscr{S}$ induced by ∇^{TX} and $\nabla^{T_{\mathbb{R}}N}$.

Still, we equip E with a Hermitian metric g^E. We recall that the connection ∇^E on E is defined by (2.2.28).

Let $g^{\mathscr{S}\otimes E}$ be the metric on $\mathscr{S}\otimes E$ induced by $g^{\mathscr{S}}$ and g^E.

Let $\nabla^{\mathscr{S}\otimes E}$ be the connection on $\mathscr{S}\otimes E$ induced by $\nabla^{\mathscr{S}}$ and ∇^E.

2.4.4. *Superconnections.*
Let $g^{\mathscr{F}}$ be the L^2-metric on $\mathscr{F} = \mathscr{C}^\infty(Y,\mathscr{S}\otimes E)$ induced by g^{TX}, g^{TN} and g^E.

Let $A^{\mathscr{F},*}$ be the adjoint superconnection of $A^{\mathscr{F}}$ (cf. [BL95, §1]).

Let $N^{\Lambda^{\cdot}(T^*S)}$ be the number operator on $\Lambda^{\cdot}(T^*S)$.

Set

$$C^{\mathscr{F}} = 2^{-N^{\Lambda^{\cdot}(T^*S)}} \left(A^{\mathscr{F},*} + A^{\mathscr{F}}\right) 2^{N^{\Lambda^{\cdot}(T^*S)}} \ . \tag{2.4.30}$$

Then $C^{\mathscr{F}}$ is still a superconnection on $\mathscr{F}$. We also define an auxiliary operator

$$D^{\mathscr{F}} = 2^{-N^{\Lambda^{\cdot}(T^*S)}} \left(A^{\mathscr{F},*} - A^{\mathscr{F}}\right) 2^{N^{\Lambda^{\cdot}(T^*S)}} \ . \tag{2.4.31}$$

Then $D^{\mathscr{F}} \in \Omega^{\cdot}(S,\mathrm{End}(\mathscr{F}))$. Moreover, we have

$$C^{\mathscr{F},2} = -D^{\mathscr{F},2} \ . \tag{2.4.32}$$

Let $\nabla^{\mathscr{F},*}$ be the adjoint connection with respect to $g^{\mathscr{F}}$. Set

$$\begin{aligned}\nabla^{\mathscr{F},\mathrm{u}} &= \frac{1}{2}\left(\nabla^{\mathscr{F},*}+\nabla^{\mathscr{F}}\right) ,\\ \omega^{\mathscr{F}} &= \nabla^{\mathscr{F},*}-\nabla^{\mathscr{F}} = \left(g^{\mathscr{F}}\right)^{-1}\nabla^{\mathscr{F}}g^{\mathscr{F}} .\end{aligned} \tag{2.4.33}$$

Then $\nabla^{\mathscr{F},\mathrm{u}}$ is a unitary connection on $\mathscr{F}$ and $\omega^{\mathscr{F}}\in\Omega^1(S,\mathrm{End}(\mathscr{F}))$.

By [BL95, Proposition 3.9], the following identities hold

$$\begin{aligned}C^{\mathscr{F}} &= \overline{\partial}_N^{E,*}+\overline{\partial}_N^{E}+d_X^{\mathscr{E},*}+d_X^{\mathscr{E}}+\nabla^{\mathscr{F},\mathrm{u}}-\frac{1}{4}c(T) ,\\ D^{\mathscr{F}} &= \overline{\partial}_N^{E,*}-\overline{\partial}_N^{E}+d_X^{\mathscr{E},*}-d_X^{\mathscr{E}}+\frac{1}{2}\omega^{\mathscr{F}}-\frac{1}{4}\hat{c}(T) .\end{aligned} \tag{2.4.34}$$

For $t,u>0$, let $C^{\mathscr{F}}_{t,u}$ (resp. $D^{\mathscr{F}}_{t,u}$) be $C^{\mathscr{F}}$ (resp. $D^{\mathscr{F}}$) with g^{TN} replaced by $\frac{1}{t}g^{TN}$ and g^{TX} replaced by $\frac{1}{u}g^{TX}$.

For convenience, we introduce the following conjugated operators

$$\begin{aligned}\mathfrak{C}^{\mathscr{F}}_{t,u} &= u^{\frac{1}{2}N^{\Lambda^{\cdot}(T^*M)}}t^{\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}}C^{\mathscr{F}}_{t,u}t^{-\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}}u^{-\frac{1}{2}N^{\Lambda^{\cdot}(T^*M)}} ,\\ \mathfrak{D}^{\mathscr{F}}_{t,u} &= u^{\frac{1}{2}N^{\Lambda^{\cdot}(T^*M)}}t^{\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}}D^{\mathscr{F}}_{t,u}t^{-\frac{1}{2}N^{\Lambda^{\cdot}(\overline{T^*N})}}u^{-\frac{1}{2}N^{\Lambda^{\cdot}(T^*M)}} .\end{aligned} \tag{2.4.35}$$

Then

$$\begin{aligned}\mathfrak{C}^{\mathscr{F}}_{t,u} &= \sqrt{t}\big(\overline{\partial}_N^{E,*}+\overline{\partial}_N^{E}\big)+\sqrt{u}\big(d_X^{\mathscr{E},*}+d_X^{\mathscr{E}}\big)+\nabla^{\mathscr{F},\mathrm{u}}-\frac{1}{4\sqrt{u}}c(T) ,\\ \mathfrak{D}^{\mathscr{F}}_{t,u} &= \sqrt{t}\big(\overline{\partial}_N^{E,*}-\overline{\partial}_N^{E}\big)+\sqrt{u}\big(d_X^{\mathscr{E},*}-d_X^{\mathscr{E}}\big)+\frac{1}{2}\omega^{\mathscr{F}}-\frac{1}{4\sqrt{u}}\hat{c}(T) .\end{aligned} \tag{2.4.36}$$

Set

$$\begin{aligned}C^{\mathscr{F}}_{\mathrm{v}} &= \overline{\partial}_N^{E,*}+\overline{\partial}_N^{E} , \quad & D^{\mathscr{F}}_{\mathrm{v}} &= \overline{\partial}_N^{E,*}-\overline{\partial}_N^{E} ,\\ C^{\mathscr{F}}_{\mathrm{h}} &= d_X^{\mathscr{E},*}+d_X^{\mathscr{E}} , \quad & D^{\mathscr{F}}_{\mathrm{h}} &= d_X^{\mathscr{E},*}-d_X^{\mathscr{E}} .\end{aligned} \tag{2.4.37}$$

Then

$$\begin{aligned}\mathfrak{C}^{\mathscr{F}}_{t,u} &= \sqrt{t}C^{\mathscr{F}}_{\mathrm{v}}+\sqrt{u}C^{\mathscr{F}}_{\mathrm{h}}+\nabla^{\mathscr{F},\mathrm{u}}-\frac{1}{4\sqrt{u}}c(T) ,\\ \mathfrak{D}^{\mathscr{F}}_{t,u} &= \sqrt{t}D^{\mathscr{F}}_{\mathrm{v}}+\sqrt{u}D^{\mathscr{F}}_{\mathrm{h}}+\frac{1}{2}\omega^{\mathscr{F}}-\frac{1}{4\sqrt{u}}\hat{c}(T) .\end{aligned} \tag{2.4.38}$$

Let $\mathfrak{C}^{\mathscr{F},[0]}_{t,u}$ (resp. $\mathfrak{D}^{\mathscr{F},[0]}_{t,u}$) be the degree zero component of $\mathfrak{C}^{\mathscr{F}}_{t,u}$ (resp. $\mathfrak{D}^{\mathscr{F}}_{t,u}$), i.e.,

$$\begin{aligned}\mathfrak{C}^{\mathscr{F},[0]}_{t,u} &= \sqrt{t}C^{\mathscr{F}}_{\mathrm{v}}+\sqrt{u}C^{\mathscr{F}}_{\mathrm{h}} ,\\ \mathfrak{D}^{\mathscr{F},[0]}_{t,u} &= \sqrt{t}D^{\mathscr{F}}_{\mathrm{v}}+\sqrt{u}D^{\mathscr{F}}_{\mathrm{h}} .\end{aligned} \tag{2.4.39}$$

Then, $\mathfrak{C}^{\mathscr{F},[0]}_{t,u}$ (resp. $\mathfrak{D}^{\mathscr{F},[0]}_{t,u}$) acting on $\mathscr{F}=\mathscr{C}^\infty(Y,\mathscr{S}\otimes E)$ is self-adjoint (resp. skew-adjoint).

Let (e_i) be an orthonormal local basis of $T_{\mathbb{R}}N$, let (e^i) be the dual basis; let (f_α) be an orthogonal local basis of TX, let (f^α) be the dual basis; let (g_α) be a basis of TS, let (g^α) be the dual basis.

In the follows, we calculate $C^{\mathscr{F}}_{\mathrm{v}}$ $D^{\mathscr{F}}_{\mathrm{v}}$, $C^{\mathscr{F}}_{\mathrm{h}}$, $D^{\mathscr{F}}_{\mathrm{h}}$, $\nabla^{\mathscr{F},u}$ and $\omega^{\mathscr{F}}$ in local coordinates.

Since $C_{\mathrm{v}}^{\mathscr{F}}$ is the classical spin^c Dirac operator on $\mathscr{E} \otimes \Lambda^{\cdot}(T^*X) = \mathscr{C}^\infty(N, \mathscr{S} \otimes E)$, we have

$$C_{\mathrm{v}}^{\mathscr{F}} = c(e_i)\nabla_{e_i}^{\mathscr{S}\otimes E} , \quad D_{\mathrm{v}}^{\mathscr{F}} = \sqrt{-1}c(Je_i)\nabla_{e_i}^{\mathscr{S}\otimes E} . \tag{2.4.40}$$

By [BL95, (3.24), (3.31)], we have

$$d_X^{\mathscr{E}} = f^\alpha \wedge \nabla_{f_\alpha}^{\mathscr{E}} , \quad d_X^{\mathscr{E},*} = -i_{f_\alpha}\nabla_{f_\alpha}^{\mathscr{E}} - i_{f_\alpha}\omega^{\mathscr{E}}(f_\alpha) . \tag{2.4.41}$$

Then

$$\begin{aligned} C_{\mathrm{h}}^{\mathscr{F}} &= c(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E}} + \frac{1}{2}c(f_\alpha)\omega^{\mathscr{E}}(f_\alpha) - \frac{1}{2}\hat{c}(f_\alpha)\omega^{\mathscr{E}}(f_\alpha) \\ &= c(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E},\mathrm{u}} - \frac{1}{2}\hat{c}(f_\alpha)\omega^{\mathscr{E}}(f_\alpha) , \\ D_{\mathrm{h}}^{\mathscr{F}} &= -\hat{c}(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E}} - \frac{1}{2}\hat{c}(f_\alpha)\omega^{\mathscr{E}}(f_\alpha) + \frac{1}{2}c(f_\alpha)\omega^{\mathscr{E}}(f_\alpha) \\ &= -\hat{c}(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E},\mathrm{u}} + \frac{1}{2}c(f_\alpha)\omega^{\mathscr{E}}(f_\alpha) . \end{aligned} \tag{2.4.42}$$

We recall that $m = \dim X$. Let $dv_X \in \Omega^m(X)$ be the volume form on X induced by g^{TX}. For U a vector field on S, set

$$k_X(U) = \big(dv_X\big)^{-1} L_{U^H} dv_X . \tag{2.4.43}$$

Then

$$k_X \in \mathscr{C}^\infty(\mathcal{N}, T^*S) \subseteq \Omega^1(S, \mathrm{End}(\mathscr{F})) . \tag{2.4.44}$$

Let $\nabla^{\Lambda^{\cdot}(T^*X)}$ be the connection on $\Lambda^{\cdot}(T^*X)$ induced by ∇^{TX}.

Let $\nabla^{\Lambda^{\cdot}(T^*X)\otimes\mathscr{E}}$ be the connection on $\Lambda^{\cdot}(T^*X) \otimes \mathscr{E}$ induced by $\nabla^{\Lambda^{\cdot}(T^*X)}$ and $\nabla^{\mathscr{E},\mathrm{u}}$.

For U a vector field on S, set

$$\omega^{\Lambda^{\cdot}(T^*X)}(U) = \big(g^{\Lambda^{\cdot}(T^*X)}\big)^{-1} L_{U^H} g^{\Lambda^{\cdot}(T^*X)} . \tag{2.4.45}$$

Then

$$\omega^{\Lambda^{\cdot}(T^*X)} \in \mathscr{C}^\infty(\mathcal{N}, T^*S \otimes \mathrm{End}(\Lambda^{\cdot}(T^*X))) \subseteq \Omega^1(S, \mathrm{End}(\mathscr{F})) . \tag{2.4.46}$$

By [BL95, (3.37)], we have

$$\begin{aligned} \nabla^{\mathscr{F},\mathrm{u}} &= g^\alpha \nabla_{g_\alpha}^{\Lambda^{\cdot}(T^*X)\otimes\mathscr{E}} + \frac{1}{2} g^\alpha k_X(g_\alpha) , \\ \omega^{\mathscr{F}} &= g^\alpha \omega^{\mathscr{E}}(g_\alpha) \otimes \mathrm{Id}_{\Lambda^{\cdot}(T^*X)} + g^\alpha \mathrm{Id}_{\mathscr{E}} \otimes \omega^{\Lambda^{\cdot}(T^*X)}(g_\alpha) + g^\alpha k_X(g_\alpha) . \end{aligned} \tag{2.4.47}$$

2.4.5. *A positive line bundle over N.*

In the sequel, we suppose that N is equipped with a line bundle L_0 and that the action of G over N lifts to L_0. Set

$$L = P \times_G L_0 . \tag{2.4.48}$$

Let L^p be the p-th tensor power of L. For $p \in \mathbb{N}$, set

$$E_p = E \otimes L^p . \tag{2.4.49}$$

We equip L with a Hermitian metric g^L. Then (L, g^L) satisfies the same properties as (E, g^E). We construct the connection ∇^L on L in the same way as ∇^E (cf. §2.2.4).

Let $R^L = \nabla^{L,2}$ be the curvature of ∇^L. We suppose that $\sqrt{-1}R^L\big|_N$ is a positive $(1,1)$-form on N. By Kodaira's vanishing theorem, this assumption implies

$$\bigoplus_{k>0} H^k(N, E_p) = 0 \tag{2.4.50}$$

for p large enough.

We define ω^L in the same way as ω^E, i.e.,

$$\omega^L = (g^L)^{-1} d_M g^L \in \mathscr{C}^\infty\big(\mathcal{N}, T^*M\big) \ . \tag{2.4.51}$$

We make the fundamental assumption that $\omega^L\big|_Y \in \mathscr{C}^\infty\big(Y, T^*X\big)$ is nowhere-zero.

By [BMaZ15, Proposition 9.15], the assumption implies

$$\chi(X) = 0 \ . \tag{2.4.52}$$

By [BMaZ15, Theorem 4.4], the assumption implies

$$H^{\cdot}(X, H^0(N, E_p)) = 0 \tag{2.4.53}$$

for p large enough. Then, by Remark 2.4.2 and (2.4.53), we get

$$H^{\cdot}_{\mathrm{tot}}(Y, E_p) = 0 \tag{2.4.54}$$

for p large enough.

We remark that the proof for (2.4.52) and (2.4.53) given in [BMaZ15] involves Toeplitz operators. We will give a more direct proof in §2.5.1.

Let g^{E_p} be the metric on E_p induced by g^E and g^L. Let ∇^{E_p} be the connection on E_p induced by ∇^E and ∇^L. All the previous results concerning (E, g^E, ∇^E) hold for $(E_p, g^{E_p}, \nabla^{E_p})$.

Let $\mathscr{E}_p$ be $\mathscr{E}$ with E replaced by E_p. Let $\mathscr{F}_p$ be $\mathscr{F}$ with E replaced by E_p.

Let $C^{\mathscr{F}_p}$ (resp. $D^{\mathscr{F}_p}$) be the $C^{\mathscr{F}}$ (resp. $D^{\mathscr{F}}$) with E replaced by E_p. These operators act on $\Omega^{\cdot}(S, \mathscr{F}_p)$.

2.4.6. *Index bundle and the associated odd characteristic forms.*

We assume that $p \in \mathbb{N}$ is large enough such that (2.4.53) holds. Set

$$H_p = q_* E_p = H^0(N, E_p) \subseteq \mathscr{E}_p \ , \tag{2.4.55}$$

which is a flat vector bundle over M. Its flat connection ∇^{H_p} is defined by (2.3.37).

Let g^{H_p} be the metric on H_p induced by $g^{\mathscr{E}_p}$.

Set

$$\mathscr{H}_p = \Omega^{\cdot}(X, H_p) \ , \tag{2.4.56}$$

which is an infinite dimensional vector bundle over S. Then ∇^{H_p} is a superconnection on $\mathscr{H}_p$ over S.

Here, we are with the same setting as [BL95, §3].

For $u > 0$, let $C_u^{\mathscr{H}_p}$ (resp. $D_u^{\mathscr{H}_p}$) be the C_{4u} (resp. D_{4u}) defined in [BL95, (3.50)] with B replaced by S and W replaced by $\mathscr{H}_p$. Then $C_u^{\mathscr{H}_p}$ is a superconnection on $\mathscr{H}_p$ over S. and $D_u^{\mathscr{H}_p} \in \Omega^{\cdot}(S, \mathrm{End}(\mathscr{H}_p))$.

Let

$$P_p : \mathscr{E}_p \to H_p \tag{2.4.57}$$

be the orthogonal projection.

We have

$$C_u^{\mathscr{H}_p} = P_p C_u^{\mathscr{F}_p} P_p\ , \quad D_u^{\mathscr{H}_p} = P_p D_u^{\mathscr{F}_p} P_p\ , \quad C_u^{\mathscr{H}_p,2} = -D_u^{\mathscr{H}_p,2}\ . \tag{2.4.58}$$

In the sequel, we denote $\varphi = (2\pi i)^{-\frac{1}{2}N^{\Lambda^\cdot(T^*S)}}$.

We equip $\mathscr{H}_p = \Omega^\cdot(X, H_p)$ with the $\mathbb{Z}_2$-grading $\Omega^{\mathrm{even/odd}}(X, H_p)$.

Set

$$\begin{aligned} \alpha_{H_p,u} &= \sqrt{2\pi i}\,\varphi\,\mathrm{Tr_s}\left[D_u^{\mathscr{H}_p}\exp\left(D_u^{\mathscr{H}_p,2}\right)\right] \in \Omega^\cdot(S)\ , \\ \beta_{H_p,u} &= \varphi\,\mathrm{Tr_s}\left[\frac{N^{\Lambda^\cdot(T^*X)}}{2}\left(1+2D_u^{\mathscr{H}_p,2}\right)\exp\left(D_u^{\mathscr{H}_p,2}\right)\right] \in \Omega^\cdot(S)\ . \end{aligned} \tag{2.4.59}$$

By [BL95, Theorem 1.8,1.9], we have

$$\alpha_{H_p,u} \in \Omega^{\mathrm{odd}}(S)\ , \quad \beta_{H_p,u} \in \Omega^{\mathrm{even}}(S)\ . \tag{2.4.60}$$

By[BL95, Theorem 1.8, 2.11], the following proposition holds.

Proposition 2.4.3. *We have*

$$d^S\alpha_{H_p,u} = 0\ , \qquad \frac{\partial}{\partial u}\alpha_{H_p,u} = \frac{1}{u}d_S\beta_{H_p,u}\ . \tag{2.4.61}$$

Still, set $f(x) = xe^{x^2}$.

By [BL95, Theorem 3.16,3.21], (2.4.52) and of (2.4.53), the following thearem holds.

Theorem 2.4.4. *The following properties hold for p large enough.*

As $u \to +\infty$, we have

$$\alpha_{H_p,u} = \mathscr{O}\big(1/\sqrt{u}\big)\ , \quad \beta_{H_p,u} = \mathscr{O}\big(1/\sqrt{u}\big)\ . \tag{2.4.62}$$

As $u \to 0$, we have

$$\alpha_{H_p,u} = \pi_*\Big[e(TX, \nabla^{TX})\, f(H_p, \nabla^{H_p}, g^{H_p})\Big] + \mathscr{O}\big(\sqrt{u}\big)\ , \quad \beta_{H_p,u} = \mathscr{O}\big(\sqrt{u}\big)\ . \tag{2.4.63}$$

Let $\mathscr{T}(T^HM, g^{TX}, g^{H_p}) \in \Omega^\cdot(S)$ be the real torsion form, defined by [BL95, Definition 3.22], associated with $\pi : M \to S$, T^HM, g^{TX}, H_p, ∇^{H_p} and g^{H_p}, i.e.,

$$\mathscr{T}(T^HM, g^{TX}, g^{H_p}) = -\int_0^\infty \beta_{H_p,u}\frac{du}{u}\ . \tag{2.4.64}$$

Theorem 2.4.5 ([BL95, Theorem 3.23])**.** *The torsion form $\mathscr{T}(T^HM, g^{TX}, g^{H_p}) \in \Omega^\cdot(S)$ is even. Moreover,*

$$d^S\mathscr{T}(T^HM, g^{TX}, g^{H_p}) = \pi_*\Big[e(TX, \nabla^{TX})\, f(H_p, \nabla^{H_p}, g^{H_p})\Big]\ . \tag{2.4.65}$$

2.4.7. *The even/odd characteristic forms and the analytic torsion form.*

In the sequel, we suppose that S is compact.

Set

$$\begin{aligned} \gamma_{\mathrm{tot},p,t,u} &= \varphi\,\mathrm{Tr_s}\left[\exp\left(-C_{t,u}^{\mathscr{F}_p,2}\right)\right] = \varphi\,\mathrm{Tr_s}\left[\exp\left(D_{t,u}^{\mathscr{F}_p,2}\right)\right] \\ &= \varphi\,\mathrm{Tr_s}\left[\exp\left(-\mathfrak{C}_{t,u}^{\mathscr{F}_p,2}\right)\right] = \varphi\,\mathrm{Tr_s}\left[\exp\left(\mathfrak{D}_{t,u}^{\mathscr{F}_p,2}\right)\right] \in \Omega^\cdot(S)\ . \end{aligned} \tag{2.4.66}$$

Proposition 2.4.6. *For any $t, u > 0$, we have*

$$\gamma_{\mathrm{tot},p,t,u} = 0\ . \tag{2.4.67}$$

Proof. The same argument as (2.3.43) implies

$$\frac{\partial}{\partial t}\gamma_{\mathrm{tot},p,t,u} = \frac{\partial}{\partial u}\gamma_{\mathrm{tot},p,t,u} = 0 \,. \tag{2.4.68}$$

Then it is sufficient to show that

$$\lim_{u\to\infty}\gamma_{\mathrm{tot},p,u,u} = 0 \,, \tag{2.4.69}$$

i.e.,

$$\lim_{u\to\infty}\mathrm{Tr_s}\Big[\exp\Big(-\mathfrak{C}^{\mathscr{F}_p,2}_{u,u}\Big)\Big] = 0 \,. \tag{2.4.70}$$

For $k\in\mathbb{N}$, let $C^{\mathscr{F}_p,2,[k]} : \Omega^{\cdot}(S,\mathscr{F}) \to \Omega^{\cdot+k}(S,\mathscr{F})$ be the degree k component of $C^{\mathscr{F}_p,2}$. Then

$$\mathfrak{C}^{\mathscr{F}_p,2}_{u,u} = \sum_{j=0}^{4} u^{1-j/2}C^{\mathscr{F}_p,2,[j]} \,. \tag{2.4.71}$$

By Hodge theory and (2.4.54), we have

$$\ker C^{\mathscr{F}_p,2,[0]} \simeq H^{\cdot}_{\mathrm{tot}}(Y,E) = 0 \,. \tag{2.4.72}$$

Then there exists $c>0$ such that

$$C^{\mathscr{F}_p,2,[0]} \geqslant c \,. \tag{2.4.73}$$

It is standard that (2.4.71) and (2.4.73) imply (2.4.70). See, for example, [BerGV04, §9]. □

Set

$$\begin{aligned}\alpha_{\mathrm{tot},p,t,u} &= \sqrt{2\pi i}\,\varphi\,\mathrm{Tr_s}\Big[D^{\mathscr{F}_p}_{t,u}\exp\Big(D^{\mathscr{F}_p,2}_{t,u}\Big)\Big] \in \Omega^{\cdot}(S) \,,\\ \beta_{\mathrm{tot},p,t,u} &= \varphi\,\mathrm{Tr_s}\left[\frac{N^{\Lambda^{\cdot}(T^*X)}}{2}\Big(1+2D^{\mathscr{F}_p,2}_{t,u}\Big)\exp\Big(D^{\mathscr{F}_p,2}_{t,u}\Big)\right] \in \Omega^{\cdot}(S) \,.\end{aligned} \tag{2.4.74}$$

By [BL95, Theorem 1.8,1.9], we have

$$\alpha_{\mathrm{tot},p,t,u} \in \Omega^{\mathrm{odd}}(S) \,, \quad \beta_{\mathrm{tot},p,t,u} \in \Omega^{\mathrm{even}}(S) \,. \tag{2.4.75}$$

Proposition 2.4.7. *We have*

$$d_S\alpha_{\mathrm{tot},p,t,u} = 0 \,, \qquad \frac{\partial}{\partial u}\alpha_{\mathrm{tot},p,t,u} = \frac{1}{u}d_S\beta_{\mathrm{tot},p,t,u} \,. \tag{2.4.76}$$

Proof. The proof is the same as Proposition 2.3.5, 2.3.6. □

We recall that $\alpha_t,\beta_t\in\Omega^{\cdot}(M)$ are defined in §2.3. Let $\alpha_{p,t},\beta_{p,t}\in\Omega^{\cdot}(M)$ be the α_t,β_t with E replaced by E_p.

We state two theorems whose proofs are delayed to §2.5.3- §2.5.5.

Theorem 2.4.8. *For $p\in\mathbb{N}$ large enough, given $t\geqslant 1$, as $u\to+\infty$,*

$$\alpha_{\mathrm{tot},p,t,u} = \mathscr{O}\big(1/\sqrt{u}\big) \,, \quad \beta_{\mathrm{tot},p,t,u} = \mathscr{O}\big(1/\sqrt{u}\big) \,. \tag{2.4.77}$$

Moreover (2.4.77) *holds uniformly in $t\geqslant 1$.*

There exists $\delta\in\left]0,\frac{1}{2}\right]$ such that given $t\geqslant 1$, as $u\to 0$, we have

$$\alpha_{\mathrm{tot},p,t,u} = \pi_*\Big[e(TX,\nabla^{TX})\,\alpha_{p,t}\Big] + \mathscr{O}\big(u^{\delta}\big) \,, \quad \beta_{\mathrm{tot},p,t,u} = \mathscr{O}\big(u^{\delta}\big) \,, \tag{2.4.78}$$

Moreover, (2.4.78) *is uniform in $t\geqslant 1$.*

Theorem 2.4.9. *For $p \in \mathbb{N}$ large enough, given $u > 0$, as $t \to +\infty$, we have*

$$\alpha_{\mathrm{tot},p,t,u} = \alpha_{H_p,u} + \mathscr{O}\big(1/\sqrt{t}\big) \ , \quad \beta_{\mathrm{tot},p,t,u} = \beta_{H_p,u} + \mathscr{O}\big(1/\sqrt{t}\big) \ . \tag{2.4.79}$$

In the sequel, we always suppose that p is large enough such that Theorem 2.4.8 and Theorem 2.4.9 hold.

Definition 2.4.10. For any $t > 0$, the analytic torsion form

$$\mathscr{T}_{\mathrm{tot},t}(T^H M, g^{TN}, g^{TX}, g^{E_p}) \in \Omega^{\mathrm{even}}(S) \tag{2.4.80}$$

is defined by

$$\mathscr{T}_{\mathrm{tot},t}(T^H M, g^{TN}, g^{TX}, g^{E_p}) = -\int_0^\infty \beta_{\mathrm{tot},p,t,u} \frac{du}{u} \ . \tag{2.4.81}$$

Let Q^S be the vector space of real even differential forms on S. Let $Q^{S,0} \subseteq Q^S$ be the vector subspace of exact real even differential forms on S, which is closed under the $\mathscr{C}^\infty$-topology. Let $Q^S/Q^{S,0}$ be the quotient space.

Let Z^S be the vector space generated by the closed chains in S. By de Rham's theorem, for any $\alpha \in Q^S$, $\alpha \in Q^{S,0}$ if and only if

$$\int_c \alpha = 0 \tag{2.4.82}$$

for any $c \in Z^S$. Thus there is a natural injection

$$Q^S/Q^{S,0} \hookrightarrow Z^{S,*} \ . \tag{2.4.83}$$

We equip $Q^S/Q^{S,0}$ with the topology such that $\alpha_t \in Q^S/Q^{S,0}$ converges to α_0 if and only if $\int_c \alpha_t$ converges to $\int_c \alpha_0$ for any $c \in Z^S$.

Theorem 2.4.11. *The form $\mathscr{T}_{\mathrm{tot},t}(g^{TN}, g^{TX}, g^{E_p}) \in \Omega^{\cdot}(S)$ is even. Moreover*

$$d_S \mathscr{T}_{\mathrm{tot},t}(T^H M, g^{TN}, g^{TX}, g^{E_p}) = \pi_*\Big[e(TX, \nabla^{TX})\, \alpha_{p,t}\Big] \ . \tag{2.4.84}$$

For $t_1, t_2 > 0$, the following identity holds in $Q^S/Q^{S,0}$,

$$\begin{aligned} &\Big[\mathscr{T}_{\mathrm{tot},t_2}(T^H M, g^{TN}, g^{TX}, g^{E_p}) - \mathscr{T}_{\mathrm{tot},t_1}(T^H M, g^{TN}, g^{TX}, g^{E_p})\Big] \\ &= \bigg[\pi_*\Big[e(TX, \nabla^{TX}) \Big(\int_{t_1}^{t_2} \beta_{p,t} \frac{dt}{t}\Big)\Big]\bigg] \ . \end{aligned} \tag{2.4.85}$$

The following identity holds in Q^S :

$$\lim_{t \to \infty} \mathscr{T}_{\mathrm{tot},t}(T^H M, g^{TN}, g^{TX}, g^{E_p}) = \mathscr{T}(T^H M, g^{TX}, g^{H_p}) \ . \tag{2.4.86}$$

The following identity holds in $Q^S/Q^{S,0}$:

$$\begin{aligned} \lim_{t \to 0} \bigg[&\mathscr{T}_{\mathrm{tot},t}(T^H M, g^{TN}, g^{TX}, g^{E_p}) \\ &- \frac{1}{2t} \pi_*\Big[e(TX, \nabla^{TX}) q_*\Big[\frac{\omega}{2\pi} \mathrm{Td}(TN, \nabla^{TN}) \mathrm{ch}(E_p, \nabla^{E_p})\Big]\Big]\bigg] \\ = \Big[\mathscr{T}(T^H M, &g^{TX}, g^{H_p}) + \pi_*\Big[e(TX, \nabla^{TX}) \mathscr{T}(g^{TN}, g^{E_p})\Big]\Big] \ . \end{aligned} \tag{2.4.87}$$

Proof. By Proposition 2.4.7, Theorem 2.4.8 and Definition 2.4.10, we get (2.4.84).

For proving (2.4.85), we apply the same transgression technique as the proof of Proposition 2.3.6. Set

$$\mathscr{N}_+ = \mathscr{N} \times \mathbb{R}_+ \,, \quad M_+ = M \times \mathbb{R}_+ \,, \quad S_+ = S \times \mathbb{R}_+ \,. \tag{2.4.88}$$

Let

$$q_+ : \mathscr{N}_+ \to M_+ \,, \quad \pi_+ : M_+ \to S_+ \tag{2.4.89}$$

be the natural extension of $q : \mathscr{N} \to M$ and $\pi : M \to S$. Let t be the coordinate on $\mathbb{R}_+$. We equip TN with the metric $\frac{1}{t} g^{TN}$. Applying (2.4.84) to the extended fibration in the same way as the proof of Proposition 2.3.6, we get

$$\left[\frac{\partial}{\partial t}\mathscr{T}_{\mathrm{tot},t}(g^{TN}, g^{TX}, g^{E_p})\right] = \left[\frac{1}{t}\pi_*\Big[e(TX,\nabla^{TX})\beta_{p,t}\Big]\right] \in Q^S/Q^{S,0} \,. \tag{2.4.90}$$

Integrating (2.4.90), we get (2.4.85).

By Theorem 2.4.8, Theorem 2.4.9 and the dominated convergence theorem, we get

$$\lim_{t\to\infty} \int_0^\infty \beta_{\mathrm{tot},p,t,u}\frac{du}{u} = \int_0^\infty \beta_{H_p,u}\frac{du}{u} \,, \tag{2.4.91}$$

which is equivalent to (2.4.86).

We recall that $g_1, g_2 \in \mathscr{C}^\infty(\mathbb{R}_+,\mathbb{R})$ are defined in §2.3.7.

By Proposition 2.3.7, (2.3.96), (2.3.97) and the fact that $\chi'(N,E_p)=0$,

$$\begin{aligned}\int_{t_1}^{t_2} \Big\{\beta_{p,t} + \frac{g_1(t)}{2}\Big(q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big] - n\chi(N,E_p)\Big) \\ + \frac{g_2(t)}{2t} q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big]\Big\}\frac{dt}{t}\end{aligned} \tag{2.4.92}$$

converges as $t_1 \to 0$ and $t_2 \to \infty$. Furthermore, by Proposition 2.2.3,

$$q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big] \in \Omega^{\cdot}(M) \tag{2.4.93}$$

is a constant 0-form on M. Then, by (2.4.52), we get

$$\begin{aligned}&\pi_*\Big[e(TX,\nabla^{TX})q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big]\Big] \\ &= \chi(X) q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big] = 0 \,.\end{aligned} \tag{2.4.94}$$

Thus

(2.4.95)

$$\begin{aligned}&\pi_*\Big[e(TX,\nabla^{TX})\Big(\int_{t_1}^{t_2}\beta_{p,t}\frac{dt}{t}\Big)\Big] \\ &\quad + \pi_*\Big[e(TX,\nabla^{TX})q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big]\Big]\int_{t_1}^{t_2}\frac{g_2(t)}{2t^2}dt \\ &= \pi_*\Big[e(TX,\nabla^{TX})\int_{t_1}^{t_2}\Big\{\beta_{p,t} + \frac{g_1(t)}{2}\Big(q_*\Big[\mathrm{Td}'(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big] - n\chi(N,E_p)\Big) \\ &\qquad + \frac{g_2(t)}{2t}q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big]\Big\}\frac{dt}{t}\Big] \,,\end{aligned}$$

which converges as $t_1 \to 0$ and $t_2 \to \infty$. Taking the limit of (2.4.95) with $t_1 \to 0$, $t_2 \to \infty$ and applying Definition 2.3.13, (2.3.98) and (2.4.85), we get

$$\begin{aligned}&\lim_{t\to 0}\Big[\mathscr{T}_{\mathrm{tot},t}(T^HM, g^{TN}, g^{TX}, g^{E_p})\\ &\qquad - \frac{1}{2t}\pi_*\Big[e(TX,\nabla^{TX})q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big]\Big]\Big]\\ &= \lim_{t\to\infty}\Big[\mathscr{T}_{\mathrm{tot},t}(T^HM, g^{TN}, g^{TX}, g^{E_p})\Big] + \Big[\pi_*\Big[e(TX,\nabla^{TX})\mathscr{T}(g^{TN}, g^{E_p})\Big]\Big]\\ &\in Q^S/Q^{S,0}\,,\end{aligned} \tag{2.4.96}$$

which, together with (2.4.86), implies (2.4.87). □

Remark 2.4.12. If the Kähler class $[\omega] \in H^{1,1}(N)$ is constant along M, by Proposition 2.2.3,

$$q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big] \tag{2.4.97}$$

is a constant function on M. Then, same as (2.4.94), we have

$$\pi_*\Big[e(TX,\nabla^{TX})q_*\Big[\frac{\omega}{2\pi}\mathrm{Td}(TN,\nabla^{TN})\mathrm{ch}(E_p,\nabla^{E_p})\Big]\Big] = 0\,. \tag{2.4.98}$$

Thus (2.4.87) is simplified as follows

$$\begin{aligned}&\lim_{t\to 0}\big[\mathscr{T}_{\mathrm{tot},t}(g^{TN}, g^{TX}, g^{E_p})\big]\\ &= \Big[\mathscr{T}(g^{TX}, g^{H_p}) + \pi_*\Big[e(TX,\nabla^{TX})\mathscr{T}(g^{TN}, g^{E_p})\Big]\Big]\,.\end{aligned} \tag{2.4.99}$$

In particular, (2.4.99) holds with the following choice of the Kähler form on N,

$$\omega = \sqrt{-1}R^L\big|_N\,. \tag{2.4.100}$$

Remark 2.4.13. If X is of odd dimension, we have

$$e(TX,\nabla^{TX}) = 0\,. \tag{2.4.101}$$

By (2.4.85) and (2.4.86) and (2.4.101), the following identity holds for $t > 0$,

$$\Big[\mathscr{T}_{\mathrm{tot},t}(T^HM, g^{TN}, g^{TX}, g^{E_p})\Big] = \Big[\mathscr{T}(T^HM, g^{TX}, g^{H_p})\Big] \in Q^S/Q^{S,0}\,. \tag{2.4.102}$$

The next section is devoted to the proofs of Theorem 2.4.8 and Theorem 2.4.9.

2.5. **Proofs of Theorem 2.4.8 and Theorem 2.4.9.**

The purpose of this section is to establish the main results of §2.4.7.

This section is organized as follows. In §2.5.1, we study the positivity of (the degree zero part of) the Levi-Civita superconnection. Some of these results were already proved in [BMaZ15] using Toeplitz operators.

In §2.5.2, we establish a Lichnerowicz formula associated with the Levi-Civita superconnection obtained in §2.4.4.

In §2.5.3, we prove Theorem 2.4.9.

In §2.5.4, we prove the $u \to \infty$ part of Theorem 2.4.8.

Finally, In §2.5.5, we establish the $u \to 0$ part of Theorem 2.4.8.

2.5.1. *Positivity of* $\mathfrak{C}_{t,u}^{\mathscr{F}_p,[0],2}$ *for* p *large enough.*

In this whole subsection, we only consider a single fiber Y together with the action of $\mathfrak{C}_{t,u}^{\mathscr{F}_p,[0],2}$ on $\Omega^{\cdot,\cdot}(Y,E_p)$. Since S is compact (cf. §2.4.7), the estimates obtained in this subsection are uniform for all fibers over S.

First, we prove a technical lemma.

We recall that $H_p = H^0(N,E_p) \subseteq \mathscr{E}_p$ is the kernel of $C_{\mathrm{v}}^{\mathscr{F}_p}$ (cf. (2.4.37)) and $P_p : \mathscr{E}_p \to H_p$ is the orthogonal projection (cf. (2.4.57)).

Let $H_p^{\perp} \subseteq \mathscr{E}_p$ be the orthogonal complement of H_p. Let

$$P_p^{\perp} : \mathscr{E}_p \to H_p^{\perp}\ . \tag{2.5.1}$$

be the orthogonal projection.

Let $\big\|\cdot\big\|$ be the L^2-norm on $\mathscr{E}_p$. Let $\big\|\cdot\big\|_{\infty}$ be the induced operator norm on $\mathrm{End}\big(\mathscr{E}_p\big)$.

Lemma 2.5.1. *For* $f \in \mathscr{C}^{\infty}(N,\mathbb{C})$, *viewed as an operator acting on* $\mathscr{E}_p$ *by multiplication, there exists* $p_0, C > 0$ *such that, for any* $p \geqslant p_0$, *we have*

$$\big\|P_p^{\perp} f P_p\big\|_{\infty} \leqslant \frac{C}{\sqrt{p}}\ . \tag{2.5.2}$$

Proof. By the proof of Kodaira's vanishing theorem (cf. [MaMar07, Theorem 1.5.6]), there exists $c>0$ such that for any $s \in H_p^{\perp}$, we have

$$\big\|C_{\mathrm{v}}^{\mathscr{F}_p} s\big\|^2 \geqslant cp\|s\|^2\ . \tag{2.5.3}$$

For $p \geqslant p_0$ and $s \in H_p$, we have

$$C_{\mathrm{v}}^{\mathscr{F}_p} P_p^{\perp} f s = C_{\mathrm{v}}^{\mathscr{F}_p} f s = \overline{\partial}_N^{E_p} f s = (\overline{\partial}_N f) s\ . \tag{2.5.4}$$

Let C be $\mathscr{C}^0$-norm of $\overline{\partial} f$. Then, by (2.5.4), we have

$$\big\|C_{\mathrm{v}}^{\mathscr{F}_p} P_p^{\perp} f s\big\| \leqslant C\|s\|\ . \tag{2.5.5}$$

By (2.5.3) and (2.5.5), for $s \in H_p$, we have

$$\big\|P_p^{\perp} f s\big\| \leqslant \frac{1}{\sqrt{cp}}\big\|C_{\mathrm{v}}^{\mathscr{F}_p} P_p^{\perp} f s\big\| \leqslant \frac{C}{\sqrt{cp}}\|s\|\ . \tag{2.5.6}$$

This proves (2.5.2). □

By (2.4.39), we have

$$\mathfrak{C}_{t,u}^{\mathscr{F}_p,[0],2} = tC_{\mathrm{v}}^{\mathscr{F}_p,2} + uC_{\mathrm{h}}^{\mathscr{F}_p,2} + \sqrt{tu}\Big[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\Big]\ . \tag{2.5.7}$$

Since g^{TN} is a fiberwise Kähler metric and $C_{\mathrm{v}}^{\mathscr{F}_p}$ is the fiberwise spin$^{\mathrm{c}}$ Dirac operator, we have

$$C_{\mathrm{v}}^{\mathscr{F}_p,2} = -\frac{1}{2}\left(\nabla_{e_i}^{\mathscr{S}\otimes E_p}\right)^2 + \frac{1}{8}r^N + \frac{1}{2}\left(R^{E_p} + \frac{1}{2}R^{\Lambda^n(TN)}\right)(e_i,e_j)\,c(e_i)c(e_j)\ , \tag{2.5.8}$$

By [BL95, Theorem 3.11], we have

$$\begin{aligned} C_{\mathrm{h}}^{\mathscr{F}_p,2} = & -\left(\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}\right)^2 + \frac{1}{4}r^X + \frac{1}{8}\left\langle f_\gamma, R^{TX}(f_\alpha,f_\beta)f_\delta\right\rangle c(f_\alpha)c(f_\beta)\hat{c}(f_\gamma)\hat{c}(f_\delta)\\ & + \frac{1}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2 + \frac{1}{8}\left(\omega^{\mathscr{E}_p}\right)^2(f_\alpha,f_\beta)\big(\hat{c}(f_\alpha)\hat{c}(f_\beta) - c(f_\alpha)c(f_\beta)\big)\\ & - \frac{1}{2}\Big[\nabla_{f_\alpha}^{\mathscr{E}_p}, \omega^{\mathscr{E}_p}(f_\beta)\Big]c(f_\alpha)\hat{c}(f_\beta)\ . \end{aligned} \tag{2.5.9}$$

Proposition 2.5.2. *The following identity holds*

$$\begin{aligned}\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right] = & -\frac{1}{2}\nabla_{e_i}^{\mathscr{S}\otimes E_p}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle\left(c(e_j)c(f_\alpha)-\sqrt{-1}c(Je_j)\hat{c}(f_\alpha)\right)\\ & -\frac{1}{2}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle\left(c(e_j)c(f_\alpha)-\sqrt{-1}c(Je_j)\hat{c}(f_\alpha)\right)\nabla_{e_i}^{\mathscr{S}\otimes E_p}\\ & +\left(R^{E_p}+\frac{1}{2}R^{\Lambda^n(TN)}\right)(e_i, f_\alpha)\left(c(e_i)c(f_\alpha)-\sqrt{-1}c(Je_i)\hat{c}(f_\alpha)\right).\end{aligned} \tag{2.5.10}$$

Proof. Since g^{TX} is constant along N, all the $c(f_\alpha)$ and $\hat{c}(f_\alpha)$ anti-commute with $C_{\mathrm{v}}^{\mathscr{F}_p}$. Then, by (2.4.42), we have

$$\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right] = c(f_\alpha)\left[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, C_{\mathrm{v}}^{\mathscr{F}_p}\right]-\frac{1}{2}\hat{c}(f_\alpha)\left[\omega^{\mathscr{E}_p}(f_\alpha), C_{\mathrm{v}}^{\mathscr{F}_p}\right]. \tag{2.5.11}$$

By Proposition 2.3.3, $C_{\mathrm{v}}^{\mathscr{F}_p}+f^\alpha\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}$ is the Levi-Civita superconnection of the infinite dimensional vector bundle $\mathscr{E}_p$ over X. Then $\left(C_{\mathrm{v}}^{\mathscr{F}_p}+f^\alpha\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}\right)^2$ is given by (2.3.75). Taking the degree 1 components in (2.3.75), we get

$$\begin{aligned}& f^\alpha\left[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, C_{\mathrm{v}}^{\mathscr{F}_p}\right]\\ & = -\frac{1}{2}\nabla_{e_i}^{\mathscr{S}\otimes E_p}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle c(e_j)f^\alpha-\frac{1}{2}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle c(e_j)f^\alpha\nabla_{e_i}^{\mathscr{S}\otimes E_p}\\ & \quad+\left(R^E+\frac{1}{2}R^{\Lambda^n(TN)}\right)(e_i, f_\alpha)c(e_i)f^\alpha.\end{aligned} \tag{2.5.12}$$

Replacing the f^α in (2.5.12) by $c(f_\alpha)$, we get

$$\begin{aligned}c(f_\alpha)\left[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, C_{\mathrm{v}}^{\mathscr{F}_p}\right] = & -\frac{1}{2}\nabla_{e_i}^{\mathscr{S}\otimes E_p}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle c(e_j)c(f_\alpha)\\ & -\frac{1}{2}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle c(e_j)c(f_\alpha)\nabla_{e_i}^{\mathscr{S}\otimes E_p}\\ & +\left(R^E+\frac{1}{2}R^{\Lambda^n(TN)}\right)(e_i, f_\alpha)c(e_i)c(f_\alpha).\end{aligned} \tag{2.5.13}$$

Since

$$\left[\overline{\partial}^{E_p}, \nabla_{f_\alpha}^{\mathscr{E}_p}\right] = \left[\overline{\partial}^{E_p,*}, \nabla_{f_\alpha}^{\mathscr{E}_p,*}\right] = 0, \tag{2.5.14}$$

we have

$$\begin{aligned}\left[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, C_{\mathrm{v}}^{\mathscr{F}_p}\right] & = \frac{1}{2}\left[\nabla_{f_\alpha}^{\mathscr{E}_p,*}+\nabla_{f_\alpha}^{\mathscr{E}_p}, \overline{\partial}^{E_p}+\overline{\partial}^{E_p,*}\right]\\ & = \frac{1}{2}\left[\nabla_{f_\alpha}^{\mathscr{E}_p,*}, \overline{\partial}^{E_p}\right]+\frac{1}{2}\left[\nabla_{f_\alpha}^{\mathscr{E}_p}, \overline{\partial}^{E_p,*}\right],\end{aligned} \tag{2.5.15}$$

and

$$\begin{aligned}\frac{1}{2}\left[\omega^{\mathscr{E}_p}(f_\alpha), C_{\mathrm{v}}^{\mathscr{F}_p}\right] & = \frac{1}{2}\left[\nabla_{f_\alpha}^{\mathscr{E}_p,*}-\nabla_{f_\alpha}^{\mathscr{E}_p}, \overline{\partial}^{E_p}+\overline{\partial}^{E_p,*}\right]\\ & = \frac{1}{2}\left[\nabla_{f_\alpha}^{\mathscr{E}_p,*}, \overline{\partial}^{E_p}\right]-\frac{1}{2}\left[\nabla_{f_\alpha}^{\mathscr{E}_p}, \overline{\partial}^{E_p,*}\right].\end{aligned} \tag{2.5.16}$$

By (2.5.15) and (2.5.16), we get

$$\frac{1}{2}\left[\omega^{\mathscr{E}_p}(f_\alpha), C_{\mathrm{v}}^{\mathscr{F}_p}\right] = (-1)^{1/2-N^{\Lambda^\cdot(\overline{T^*N})}/2}\left[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, C_{\mathrm{v}}^{\mathscr{F}_p}\right](-1)^{N^{\Lambda^\cdot(\overline{T^*N})}/2}. \tag{2.5.17}$$

By replacing the f^α in (2.5.12) by $\hat{c}(f_\alpha)$ and applying (2.5.17), we get

$$\begin{aligned}\frac{1}{2}\hat{c}(f_\alpha)\Big[\omega^{\mathscr{E}_p}_{f_\alpha}, C^{\mathscr{F}_p}_{\mathrm{v}}\Big] = {} & \frac{\sqrt{-1}}{2}\nabla^{\mathscr{S}\otimes E_p}_{e_i}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle c(Je_j)\hat{c}(f_\alpha)\\ & + \frac{\sqrt{-1}}{2}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle c(Je_j)\hat{c}(f_\alpha)\nabla^{\mathscr{S}\otimes E_p}_{e_i}\\ & - \sqrt{-1}\left(R^E + \frac{1}{2}R^{\Lambda^n(TN)}\right)(e_i, f_\alpha)c(Je_i)\hat{c}(f_\alpha)\,.\end{aligned} \tag{2.5.18}$$

By (2.5.11), (2.5.13) and (2.5.18), we get (2.5.10). □

Set

$$\begin{aligned} A_1 &= P_p\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u}P_p\,, \quad A_2 = P_p\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u}P_p^\perp\,,\\ A_3 &= P_p^\perp\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u}P_p\,, \quad A_4 = P_p^\perp\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u}P_p^\perp\,.\end{aligned} \tag{2.5.19}$$

Then

$$\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u} = \begin{pmatrix} A_1 & A_2\\ A_3 & A_4\end{pmatrix}\,. \tag{2.5.20}$$

Theorem 2.5.3. *There exist $c, C > 0$ and $p_0 > 0$ such that, for $p \geqslant p_0$ and $t, u > 0$, we have*

$$A_1 \geqslant cup^2\,, \quad A_4 \geqslant cup^2 + ctp\,, \tag{2.5.21}$$

for $s_1 \in \Omega^\cdot(M, H_p)$ and $s_2 \in \Omega^\cdot(M, H_p^\perp)$, we have

$$\begin{aligned}&\left|\langle s_1, A_2 s_2\rangle\right| = \left|\langle A_3 s_1, s_2\rangle\right|\\ &\leqslant C\sqrt{up}\sqrt{\langle A_1 s_1, s_1\rangle}\|s_2\| + C(\sqrt{tu}+u)p\|s_1\|\|s_2\|\,.\end{aligned} \tag{2.5.22}$$

Moreover, there exist $c, p_0 > 0$ such that, for $p \geqslant p_0$ and $t, u > 0$, we have

$$\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u} \geqslant cup^2\,. \tag{2.5.23}$$

Proof. In the whole proof, $c > 0$ is a small enough constant, $C > 0$ is a large enough constant, and p is always supposed to be large enough.

Step 1. We establish the positivity of $tC^{\mathscr{F}_p,2}_{\mathrm{v}}$ and $uC^{\mathscr{F}_p,2}_{\mathrm{h}}$.

By (2.5.3), we get

$$tP_p^\perp C^{\mathscr{F}_p,2}_{\mathrm{v}}P_p^\perp \geqslant ctp\,. \tag{2.5.24}$$

By (2.5.9), $C^{\mathscr{F}_p,2}_{\mathrm{h}}$ consists of a connection Laplacian and zero order terms, which are polynomials on p. Furthermore, the only term of degree$\geqslant 2$ on p is

$$\frac{p^2}{4}\left(\omega^L(f_\alpha)\right)^2\,, \tag{2.5.25}$$

which comes from

$$\frac{1}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2 = \frac{1}{4}\left(\omega^{\mathscr{E}}(f_\alpha) + p\omega^L(f_\alpha)\right)^2\,. \tag{2.5.26}$$

By the non degeneration of ω^L (cf. §2.4.5), we have

$$\frac{p^2}{4}\left(\omega^L(f_\alpha)\right)^2 \geqslant cp^2\,. \tag{2.5.27}$$

Thus the zero order part of $C_{\mathrm{h}}^{\mathscr{F}_p,2}$ is controlled from below by cp^2. Hence,

$$uC_{\mathrm{h}}^{\mathscr{F}_p,2} \geqslant cup^2 \ . \tag{2.5.28}$$

As a consequence,

$$uP_pC_{\mathrm{h}}^{\mathscr{F}_p,2}P_p \geqslant cup^2 \ , \quad uP_p^{\perp}C_{\mathrm{h}}^{\mathscr{F}_p,2}P_p^{\perp} \geqslant cup^2 \ . \tag{2.5.29}$$

Step 2. We establish a lower bound for $\sqrt{tu}\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right]$.

By (2.5.10), $\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right]$ consists of the following first order terms

$$\begin{aligned} & -\frac{1}{2}\nabla_{e_i}^{\mathscr{S}\otimes E_p}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle\Big(c(e_j)c(f_\alpha)-\sqrt{-1}c(Je_j)\hat{c}(f_\alpha)\Big) \\ & -\frac{1}{2}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle\Big(c(e_j)c(f_\alpha)-\sqrt{-1}c(Je_j)\hat{c}(f_\alpha)\Big)\nabla_{e_i}^{\mathscr{S}\otimes E_p} \ , \end{aligned} \tag{2.5.30}$$

and zero order terms, which are polynomials on p of degree$\leqslant 1$.

The zero order terms are controlled from below by $-Cp$. It rests to control the first order terms. Since $\Big(c(e_j)c(f_\alpha)-\sqrt{-1}c(Je_j)\hat{c}(f_\alpha)\Big)$ is skew-adjoint, for any $\epsilon>0$,

$$\nabla_{\cdot}^{\mathscr{S}\otimes E_p} + \frac{1}{\epsilon}\left\langle S^{TN}(\cdot)e_j, f_\alpha\right\rangle\Big(c(e_j)c(f_\alpha)-\sqrt{-1}c(Je_j)\hat{c}(f_\alpha)\Big) \tag{2.5.31}$$

is a unitary connection. Thus

$$-\epsilon\left(\nabla_{e_i}^{\mathscr{S}\otimes E_p} + \frac{1}{\epsilon}\left\langle S^{TN}(e_i)e_j, f_\alpha\right\rangle\Big(c(e_j)c(f_\alpha)-\sqrt{-1}c(Je_j)\hat{c}(f_\alpha)\Big)\right)^2 \geqslant 0 \ . \tag{2.5.32}$$

Comparing (2.5.30) and (2.5.32), we see that (2.5.30) is controlled from below by

$$\epsilon\left(\nabla_{e_i}^{\mathscr{S}\otimes E_p}\right)^2 - \frac{C}{\epsilon} \ . \tag{2.5.33}$$

Combing the lower bounds obtained for the zero order and first order parts of $\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right]$, we get

$$\sqrt{tu}\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right] \geqslant \epsilon\sqrt{tu}\left(\nabla_{e_i}^{\mathscr{S}\otimes E_p}\right)^2 - \frac{C\sqrt{tu}}{\epsilon} - C\sqrt{tu}p \ . \tag{2.5.34}$$

Replacing ϵ by $\frac{\varepsilon}{2}\sqrt{t}/\sqrt{u}$, we get

$$\sqrt{tu}\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right] \geqslant \frac{\varepsilon t}{2}\left(\nabla_{e_i}^{\mathscr{S}\otimes E_p}\right)^2 - \frac{Cu}{\varepsilon} - C\sqrt{tu}p \ . \tag{2.5.35}$$

Applying (2.5.8), we get

$$\sqrt{tu}\left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p}\right] \geqslant -\varepsilon tC_{\mathrm{v}}^{\mathscr{F}_p,2} - \varepsilon Ctp - \frac{Cu}{\varepsilon} - C\sqrt{tu}p \ . \tag{2.5.36}$$

Step 3. We prove (2.5.21).
Since

$$C_{\mathrm{v}}^{\mathscr{F}_p}P_p = P_pC_{\mathrm{v}}^{\mathscr{F}_p} = 0 \ , \tag{2.5.37}$$

we have

$$A_1 = P_p\left(\sqrt{t}C_{\mathrm{v}}^{\mathscr{F}_p} + \sqrt{u}C_{\mathrm{h}}^{\mathscr{F}_p}\right)^2 P_p = uP_pC_{\mathrm{h}}^{\mathscr{F}_p,2}P_p \ . \tag{2.5.38}$$

The first inequality in (2.5.21) follows from (2.5.29) and (2.5.38).

By (2.5.24), (2.5.29) and (2.5.36), we have

$$\begin{aligned} A_4 &= uP_p^{\perp} C_{\mathrm{h}}^{\mathscr{F}_p,2} P_p^{\perp} + (1-\varepsilon) t P_p^{\perp} C_{\mathrm{v}}^{\mathscr{F}_p,2} P_p^{\perp} \\ &\quad + P_p^{\perp} \left(\varepsilon t C_{\mathrm{v}}^{\mathscr{F}_p,2} + \sqrt{tu} \left[C_{\mathrm{v}}^{\mathscr{F}_p}, C_{\mathrm{h}}^{\mathscr{F}_p} \right] \right) P_p^{\perp} \\ &\geqslant cup^2 + (1-\varepsilon) ctp - \varepsilon Ctp - \frac{Cu}{\varepsilon} - C\sqrt{tu}p \,. \end{aligned} \tag{2.5.39}$$

By choosing ε small enough in (2.5.39) and applying the Cauchy-Schwarz inequality, we get the second inequality in (2.5.21).

Step 4. We estimate $\sqrt{tu} P_p^{\perp} C_{\mathrm{v}}^{\mathscr{F}_p} C_{\mathrm{h}}^{\mathscr{F}_p} P_p$.

For $s_1 \in \Omega^{\cdot}(X, H_p)$, we have

$$P_p^{\perp} C_{\mathrm{v}}^{\mathscr{F}_p} C_{\mathrm{h}}^{\mathscr{F}_p} s_1 = P_p^{\perp} \big(\overline{\partial}_N^{E_p} + \overline{\partial}_N^{E_p,*} \big) \big(d_X^{\mathscr{E}_p} + d_X^{\mathscr{E}_p,*} \big) s_1 \,. \tag{2.5.40}$$

We recall that $d_X^{\mathscr{E}_p}$ is the de Rham operator on $\Omega^{\cdot}(X, \mathscr{E}_p)$, which preserves $H_p = \ker \big(\overline{\partial}_N^{E_p} + \overline{\partial}_N^{E_p,*} \big)$, we have

$$\big(\overline{\partial}_N^{E_p} + \overline{\partial}_N^{E_p,*} \big) d_X^{\mathscr{E}_p} s_1 = 0 \,. \tag{2.5.41}$$

Since $d_X^{\mathscr{E}_p,*} s_1 \in \Omega^{\cdot,0}(Y, E_p)$ and $\overline{\partial}_N^{E_p,*} : \Omega^{\cdot,\cdot}(Y, E_p) \to \Omega^{\cdot,\cdot-1}(Y, E_p)$, we have

$$\overline{\partial}_N^{E_p,*} d_X^{\mathscr{E}_p,*} s_1 = 0 \,. \tag{2.5.42}$$

By (2.5.40)-(2.5.42), we get

$$P_p^{\perp} C_{\mathrm{v}}^{\mathscr{F}_p} C_{\mathrm{h}}^{\mathscr{F}_p} s_1 = P_p^{\perp} \overline{\partial}_N^{E_p} d_X^{\mathscr{E}_p,*} s_1 \,. \tag{2.5.43}$$

By (2.4.41) and (2.5.43), we get

$$P_p^{\perp} C_{\mathrm{v}}^{\mathscr{F}_p} C_{\mathrm{h}}^{\mathscr{F}_p} s_1 = i_{f_\alpha} P_p^{\perp} \overline{\partial}_N^{E_p} \nabla_{f_\alpha}^{\mathscr{E}_p} s_1 + i_{f_\alpha} P_p^{\perp} \overline{\partial}_N^{E_p} \omega^{\mathscr{E}_p}(f_\alpha) s_1 \,. \tag{2.5.44}$$

We recall that $\nabla^{\mathscr{E}_p}$ is the flat connection on $\mathscr{E}_p$, which preserves $H_p \subseteq \ker \overline{\partial}_N^{E_p}$, we have

$$\overline{\partial}_N^{E_p} \nabla_{f_\alpha}^{\mathscr{E}_p} s_1 = 0 \,. \tag{2.5.45}$$

By (2.5.44) and (2.5.45), we get

$$\begin{aligned} P_p^{\perp} C_{\mathrm{v}}^{\mathscr{F}_p} C_{\mathrm{h}}^{\mathscr{F}_p} s_1 &= i_{f_\alpha} P_p^{\perp} \overline{\partial}_N^{E_p} \omega^{\mathscr{E}_p}(f_\alpha) s_1 \\ &= i_{f_\alpha} P_p^{\perp} \overline{\partial}_N^{E_p} \big(\omega^{\mathscr{E}}(f_\alpha) + p\omega^L(f_\alpha) \big) s_1 \\ &= i_{f_\alpha} P_p^{\perp} \big(\overline{\partial}_N^{E} \omega^{\mathscr{E}}(f_\alpha) + p\overline{\partial}_N \omega^L(f_\alpha) \big) s_1 \,. \end{aligned} \tag{2.5.46}$$

Thus

$$\left\| \sqrt{tu} P_p^{\perp} C_{\mathrm{v}}^{\mathscr{F}_p} C_{\mathrm{h}}^{\mathscr{F}_p} s_1 \right\| \leqslant C\sqrt{tu}p \|s_1\| \,. \tag{2.5.47}$$

Step 5. We estimate $uP_p^{\perp} C_{\mathrm{h}}^{\mathscr{F},2} P_p$.

By (2.3.33) and (2.5.9), we get

$$\begin{aligned} C_{\mathrm{h}}^{\mathscr{F}_p,2} &= -\left(\nabla_{f_\alpha}^{\mathscr{E}_p} + \frac{1}{2}\omega^{\mathscr{E}_p}(f_\alpha) \right)^2 + \frac{1}{4} \left(\omega^{\mathscr{E}_p}(f_\alpha) \right)^2 + \Theta' \\ &= -\left(\nabla_{f_\alpha}^{\mathscr{E}_p} \right)^2 - \omega^{\mathscr{E}_p}(f_\alpha) \nabla_{f_\alpha}^{\mathscr{E}_p} + \Theta \,, \end{aligned} \tag{2.5.48}$$

where Θ and Θ' are zero order operators bounded by Cp. Since $\nabla^{\mathscr{E}_p}$ preserves $H_p = \ker P_p^\perp$, we get

$$P_p^\perp C_{\mathrm{h}}^{\mathscr{F},2} s_1 = -P_p^\perp \omega^{\mathscr{E}_p}(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E}_p} s_1 + P_p^\perp \Theta s_1 \\ = -pP_p^\perp \omega^L(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E}_p} s_1 - P_p^\perp \omega^{\mathscr{E}}(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E}_p} s_1 + P_p^\perp \Theta s_1 \,. \tag{2.5.49}$$

Applying Lemma 2.5.1 to $P_p^\perp \omega^L(f_\alpha)\nabla_{f_\alpha}^{\mathscr{E}_p} s_1$ in (2.5.49), we get

$$\left\| P_p^\perp C_{\mathrm{h}}^{\mathscr{F},2} s_1 \right\| \leqslant C\sqrt{p}\left\| \nabla_{f_\alpha}^{\mathscr{E}_p} s_1 \right\| + Cp\|s_1\| \,. \tag{2.5.50}$$

By the Cauchy-Schwarz inequality and (2.3.33), we get

$$\left\| \nabla_{f_\alpha}^{\mathscr{E}_p} s_1 \right\|^2 = \left\| \nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}} s_1 - \frac{1}{2}\omega^{\mathscr{E}_p}(f_\alpha) s_1 \right\|^2 \\ \leqslant 2\left\| \nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}} s_1 \right\|^2 + 2\left\| \frac{1}{2}\omega^{\mathscr{E}_p}(f_\alpha) s_1 \right\|^2 = 2\left\langle -\left(\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}\right)^2 s_1 + \frac{1}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2 s_1, s_1 \right\rangle \,. \tag{2.5.51}$$

Comparing $-\left(\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}\right)^2 s_1 + \frac{1}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2$ with (2.5.9), we see that

$$C_{\mathrm{h}}^{\mathscr{F}_p,2} = -\left(\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}\right)^2 s_1 + \frac{1}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2 + \Theta' \,, \tag{2.5.52}$$

with Θ' a zero order operator bounded by Cp. Then the same argument as Step 1 yields

$$-\left(\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}\right)^2 s_1 + \frac{1}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2 \leqslant C C_{\mathrm{h}}^{\mathscr{F}_p,2} \,. \tag{2.5.53}$$

By (2.5.50), (2.5.51) and (2.5.53), we get

$$\left\| P_p^\perp C_{\mathrm{h}}^{\mathscr{F}_p,2} s_1 \right\| \leqslant C\sqrt{p}\sqrt{\left\langle C_{\mathrm{h}}^{\mathscr{F}_p,2} s_1, s_1 \right\rangle} + Cp\|s_1\| \,. \tag{2.5.54}$$

Then, by (2.5.38), we get

$$\left\| uP_p^\perp C_{\mathrm{h}}^{\mathscr{F}_p,2} s_1 \right\| \leqslant C\sqrt{up}\sqrt{\langle A_1 s_1, s_1\rangle} + Cup\|s_1\| \,. \tag{2.5.55}$$

Step 6. We prove (2.5.22).

Since $\mathfrak{C}_{t,u}^{\mathscr{F}_p,[0],2}$ is self-adjoint, we get the equality in (2.5.22). We turn to prove the inequality in (2.5.22).

By (2.5.37), we have

$$A_3 s_1 = \sqrt{tu}P_p^\perp C_{\mathrm{v}}^{\mathscr{F}_p} C_{\mathrm{h}}^{\mathscr{F}_p} s_1 + uP_p^\perp C_{\mathrm{h}}^{\mathscr{F},2} s_1 \,. \tag{2.5.56}$$

Then, by (2.5.47) and (2.5.55), we get the inequality in (2.5.22).

Step 7. We prove (2.5.23).

For $s \in \mathscr{F}_p$, we have the decomposition

$$s = s_1 + s_2 \,, \tag{2.5.57}$$

with $s_1 \in \Omega^{\cdot}(X, H_p)$ and $s_2 \in \Omega^{\cdot}(X, H_p^\perp)$. Then

$$\left\langle \mathfrak{C}_{t,u}^{\mathscr{F}_p,[0],2} s, s \right\rangle = \left\langle A_1 s_1, s_1 \right\rangle + \left\langle A_4 s_2, s_2 \right\rangle + \left\langle A_2 s_2, s_1 \right\rangle + \left\langle A_3 s_1, s_2 \right\rangle \,. \tag{2.5.58}$$

By the Cauchy-Schwarz inequality and (2.5.22), for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\left|\left\langle A_2 s_2, s_1\right\rangle\right| + \left|\left\langle A_3 s_1, s_2\right\rangle\right| \\
&\leqslant C\sqrt{up}\sqrt{\langle A_1 s_1, s_1\rangle}\|s_2\| + C\sqrt{tu}p\|s_1\|\|s_2\| + Cup\|s_1\|\|s_2\| \\
&\leqslant \frac{1}{2}\Big(\left\langle A_1 s_1, s_1\right\rangle + C^2 up\|s_2\|^2\Big) + \frac{Cp}{2}\Big(\frac{u}{\varepsilon}\|s_1\|^2 + \varepsilon t\|s_2\|^2\Big) \\
&\quad + \frac{Cup}{2}\Big(\|s_1\|^2 + \|s_2\|^2\Big) \, .
\end{aligned} \tag{2.5.59}$$

By (2.5.21), (2.5.58) and (2.5.59), we get

$$\begin{aligned}
\Big\langle \mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u} s, s\Big\rangle \geqslant \Big(\frac{c}{2}p^2 - \frac{C}{2\varepsilon}p - \frac{C}{2}p\Big)u\|s_1\|^2 + \Big(cp^2 - \frac{C^2}{2}p - \frac{C}{2}p\Big)u\|s_2\|^2 \\
+ \Big(cp - \frac{\varepsilon C}{2}p\Big)t\|s_2\|^2 \, .
\end{aligned} \tag{2.5.60}$$

By choosing ε small enough, we get (2.5.23). □

Corollary 2.5.4. *We have*

$$H^{\cdot}(X, H^0(N, E_p)) = H^{\cdot}_{\mathrm{tot}}(Y, E_p) = 0 \, . \tag{2.5.61}$$

Proof. The first equality in (2.5.61) comes from Remark 2.4.2.

By Hodge theory and (2.5.23), we have

$$H^{\cdot}_{\mathrm{tot}}(Y, E_p) \simeq \ker\Big(\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u}\Big) = 0 \, . \tag{2.5.62}$$

Thus we get the second equality in (2.5.61). □

Corollary 2.5.5. *There exists $p_0 > 0$ such that, for $p \geqslant p_0$ and $t, u > 0$, we have*

$$\mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u} \geqslant \frac{t}{2}C^{\mathscr{F}_p,2}_{\mathrm{v}} + \frac{u}{2}C^{\mathscr{F}_p,2}_{\mathrm{h}} \, , \qquad \mathfrak{C}^{\mathscr{F}_p,[0],2}_{t,u} \leqslant \frac{3t}{2}C^{\mathscr{F}_p,2}_{\mathrm{v}} + \frac{3u}{2}C^{\mathscr{F}_p,2}_{\mathrm{h}} \tag{2.5.63}$$

Proof. In the proof of Proposition 2.5.3, we showed that for p large enough,

$$tC^{\mathscr{F}_p,2}_{\mathrm{v}} + uC^{\mathscr{F}_p,2}_{\mathrm{h}} + \sqrt{tu}\Big[C^{\mathscr{F}_p}_{\mathrm{v}}, C^{\mathscr{F}_p}_{\mathrm{h}}\Big] \geqslant cup^2 \, . \tag{2.5.64}$$

In fact, the argument leading to Proposition 2.5.3 could be used to show a stronger inequality: for $a > 0$, $b \in \mathbb{R}$, there exist $c_{a,b} > 0$, $p_{a,b} > 0$ such that for $p \geqslant p_{a,b}$, we have

$$atC^{\mathscr{F}_p,2}_{\mathrm{v}} + auC^{\mathscr{F}_p,2}_{\mathrm{h}} + b\sqrt{tu}\Big[C^{\mathscr{F}_p}_{\mathrm{v}}, C^{\mathscr{F}_p}_{\mathrm{h}}\Big] \geqslant c_{a,b}up^2 \, . \tag{2.5.65}$$

In particular, the following inequality holds for p large enough,

$$\frac{t}{2}C^{\mathscr{F}_p,2}_{\mathrm{v}} + \frac{u}{2}C^{\mathscr{F}_p,2}_{\mathrm{h}} \pm \sqrt{tu}\Big[C^{\mathscr{F}_p}_{\mathrm{v}}, C^{\mathscr{F}_p}_{\mathrm{h}}\Big] \geqslant 0 \, . \tag{2.5.66}$$

The '+' case is equivalent to the first inequality in (2.5.63). The '−' case is equivalent to the second inequality in (2.5.63). □

We recall that $C^{\mathscr{H}_p}_u$ is defined in §2.4.6. Let $C^{\mathscr{H}_p,[0]} \in \mathrm{End}(\mathscr{H}_p)$ be the degree zero component of $C^{\mathscr{H}_p}_1$. Then $C^{\mathscr{H}_p,[0]}$ is self-adjoint.

The following proposition is proved by Bismut-Ma-Zhang [BMaZ15, Theorem 4.4]. Hereby, we give a different proof.

Proposition 2.5.6. *There exist $c > 0$ and $p_0 > 0$ such that, for $p \geqslant p_0$, we have*

$$C^{\mathscr{H}_p,[0],2} \geqslant cp^2 \, . \tag{2.5.67}$$

Proof. For $s \in \mathscr{H}_p = \Omega^{\cdot}(X, H_p)$, by (2.4.39) and (2.4.58), we have

$$C^{\mathscr{F}_p,[0]}s - C^{\mathscr{H}_p,[0]}s = P_p^{\perp}C^{\mathscr{F}_p,[0]}s = P_p^{\perp}C_{\mathrm{h}}^{\mathscr{F}_p}s \,. \tag{2.5.68}$$

By Theorem 2.5.3, there exists $c > 0$ such that, for p large enough, we have

$$\big\|C^{\mathscr{F}_p,[0]}s\big\| \geqslant cp\big\|s\big\| \,. \tag{2.5.69}$$

Since $\nabla^{\mathscr{E}_p}$ preserves H_p, by (2.4.42), we have

$$\begin{aligned} P_p^{\perp}C_{\mathrm{h}}^{\mathscr{F}_p}s &= \frac{1}{2}\big(c(f_\alpha) - \hat{c}(f_\alpha)\big)P_p^{\perp}\omega^{\mathscr{E}_p}(f_\alpha)s \\ &= \frac{1}{2}\big(c(f_\alpha) - \hat{c}(f_\alpha)\big)P_p^{\perp}\big(\omega^{\mathscr{E}}(f_\alpha) + p\omega^{L}(f_\alpha)\big)s \,. \end{aligned} \tag{2.5.70}$$

By Lemma 2.5.1 and (2.5.70), there exists $C > 0$ such that, for p large enough, we have

$$\Big\|P_p^{\perp}C_{\mathrm{h}}^{\mathscr{F}_p}s\Big\| \leqslant C\sqrt{p}\big\|s\big\| \,. \tag{2.5.71}$$

By (2.5.68), (2.5.69) and (2.5.71), there exists $c > 0$ such that, for p large enough, we have

$$\big\|C^{\mathscr{H}_p,[0]}s\big\| \geqslant cp\big\|s\big\| \tag{2.5.72}$$

This is equivalent to (2.5.67). $\square$

2.5.2. *A Lichnerowicz formula for* $\mathfrak{D}_{t,u}^{\mathscr{F}_p,2} + z\mathfrak{D}_{t,u}^{\mathscr{F}_p}$.

Let z be an additional odd Grassmannian variable such that $z^2 = 0$.

We recall that $\big\langle S^{TX}(\cdot)\cdot,\cdot\big\rangle$ is constructed in §2.1.4.

Theorem 2.5.7. *The following identity holds*

$$\begin{aligned} &\mathfrak{D}_{t,u}^{\mathscr{F}_p,2} + z\mathfrak{D}_{t,u}^{\mathscr{F}_p} \\ &= \Big(\sqrt{t}D_{\mathrm{v}}^{\mathscr{F}_p} + \frac{\sqrt{u}}{2}c(f_\alpha)\omega^{\mathscr{E}_p}(f_\alpha) + \frac{1}{2}g^\beta\omega^{\mathscr{E}_p}(g_\beta)\Big)^2 \\ &+ z\Big(\sqrt{t}D_{\mathrm{v}}^{\mathscr{F}_p} + \frac{\sqrt{u}}{2}c(f_\alpha)\omega^{\mathscr{E}_p}(f_\alpha) + \frac{1}{2}g^\beta\omega^{\mathscr{E}_p}(g_\beta)\Big) \\ &+ u\Big(\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}} + \frac{1}{2\sqrt{u}}\big\langle S^{TX}(f_\alpha)f_\beta, g_\gamma\big\rangle c(f_\beta)g^\gamma \\ &\qquad + \frac{1}{4u}\big\langle S^{TX}(f_\alpha)g_\beta, g_\gamma\big\rangle g^\beta g^\gamma - \frac{z}{2\sqrt{u}}\hat{c}(f_\alpha)\Big)^2 \\ &- \frac{u}{8}\big\langle f_\gamma, R^{TX}(f_\alpha, f_\beta)f_\delta\big\rangle \hat{c}(f_\gamma)\hat{c}(f_\delta)c(f_\alpha)c(f_\beta) \\ &- \frac{\sqrt{u}}{4}\big\langle f_\gamma, R^{TX}(f_\alpha, g_\beta)f_\delta\big\rangle \hat{c}(f_\gamma)\hat{c}(f_\delta)c(f_\alpha)g^\beta \\ &- \frac{1}{8}\big\langle f_\gamma, R^{TX}(g_\alpha, g_\beta)f_\delta\big\rangle \hat{c}(f_\gamma)\hat{c}(f_\delta)g^\alpha g^\beta \\ &- \frac{u}{4}r^X - \frac{u}{8}\big(\omega^{\mathscr{E}_p}\big)^2(f_\alpha, f_\beta)\hat{c}(f_\alpha)\hat{c}(f_\beta) + \frac{u}{2}\Big[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, \omega^{\mathscr{E}_p}(f_\beta)\Big]c(f_\alpha)\hat{c}(f_\beta) \\ &+ \frac{\sqrt{u}}{2}\Big[\nabla_{g_\alpha}^{\mathscr{E}_p,\mathrm{u}}, \omega^{\mathscr{E}_p}(f_\beta)\Big]g^\alpha\hat{c}(f_\beta) - \sqrt{u}\,\hat{c}(f_\alpha)\Big[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, \sqrt{t}D_{\mathrm{v}}^{\mathscr{F}_p}\Big] \,. \end{aligned} \tag{2.5.73}$$

Proof. Since

$$\Big[\hat{c}(T), D_{\mathrm{v}}^{\mathscr{F}_p}\Big] = 0\,, \tag{2.5.74}$$

by (2.4.38), we have

$$\begin{aligned}
&\mathfrak{D}_{t,u}^{\mathscr{F}_p,2} + z\mathfrak{D}_{t,u}^{\mathscr{F}_p}\\
&= tD_{\mathrm{v}}^{\mathscr{F}_p,2} + \sqrt{tu}\Big[D_{\mathrm{h}}^{\mathscr{F}_p}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big] + \sqrt{t}\Big[\frac{1}{2}\omega^{\mathscr{F}_p}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big] + z\sqrt{t}D_{\mathrm{v}}^{\mathscr{F}_p}\\
&+ \Big(\sqrt{u}D_{\mathrm{h}}^{\mathscr{F}_p} + \frac{1}{2}\omega^{\mathscr{F}_p} - \frac{1}{4\sqrt{u}}\hat{c}(T)\Big)^2 + z\Big(\sqrt{u}D_{\mathrm{h}}^{\mathscr{F}_p} + \frac{1}{2}\omega^{\mathscr{F}_p} - \frac{1}{4\sqrt{u}}\hat{c}(T)\Big)\,.
\end{aligned} \tag{2.5.75}$$

By (2.4.42), we have

$$\sqrt{tu}\Big[D_{\mathrm{h}}^{\mathscr{F}_p}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big] = \Big[\frac{\sqrt{u}}{2}c(f_\alpha)\omega^{\mathscr{E}_p}(f_\alpha), \sqrt{t}D_{\mathrm{v}}^{\mathscr{F}_p}\Big] - \sqrt{tu}\,\hat{c}(f_\alpha)\Big[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big]\,. \tag{2.5.76}$$

By (2.4.47) and

$$\Big[\omega^{\Lambda^{\cdot}(T^*X)}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big] = \Big[k_X, D_{\mathrm{v}}^{\mathscr{F}_p}\Big] = 0\,, \tag{2.5.77}$$

we have

$$\sqrt{t}\Big[\frac{1}{2}\omega^{\mathscr{F}_p}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big] = \Big[\frac{1}{2}g^\alpha\omega^{\mathscr{E}_p}(g_\alpha), \sqrt{t}D_{\mathrm{v}}^{\mathscr{F}_p}\Big]\,. \tag{2.5.78}$$

By (2.5.76) and (2.5.78), we have

$$\begin{aligned}
&tD_{\mathrm{v}}^{\mathscr{F}_p,2} + \sqrt{tu}\Big[D_{\mathrm{h}}^{\mathscr{F}_p}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big] + \sqrt{u}\Big[\frac{1}{2}\omega^{\mathscr{F}_p}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big]\\
&= \Big(\sqrt{t}D_{\mathrm{v}}^{\mathscr{F}_p} + \frac{\sqrt{u}}{2}c(f_\alpha)\omega^{\mathscr{E}_p}(f_\alpha) + \frac{1}{2}g^\beta\omega^{\mathscr{E}_p}(g_\beta)\Big)^2 + \frac{u}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2\\
&- \frac{u}{8}\left(\omega^{\mathscr{E}_p}\right)^2(f_\alpha, f_\beta)c(f_\alpha)c(f_\beta) - \frac{\sqrt{u}}{4}\left(\omega^{\mathscr{E}_p}\right)^2(f_\alpha, g_\beta)c(f_\alpha)g^\beta\\
&- \frac{1}{8}\left(\omega^{\mathscr{E}_p}\right)^2(g_\alpha, g_\beta)g^\alpha g^\beta - \sqrt{tu}\,\hat{c}(f_\alpha)\Big[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}}, D_{\mathrm{v}}^{\mathscr{F}_p}\Big]
\end{aligned} \tag{2.5.79}$$

Applying [BL95, Theorem 3.11] with F replaced by $\mathscr{E}_p$, we get

$$
\begin{aligned}
&\Big(\sqrt{u}D_{\mathrm{h}}^{\mathscr{F}_p}+\frac{1}{2}\omega^{\mathscr{F}_p}-\frac{1}{4\sqrt{u}}\hat{c}(T)\Big)^2+z\Big(\sqrt{u}D_{\mathrm{h}}^{\mathscr{F}_p}+\frac{1}{2}\omega^{\mathscr{F}_p}-\frac{1}{4\sqrt{u}}\hat{c}(T)\Big)\\
&=u\Big(\nabla_{f_\alpha}^{\mathscr{E}_p}+\frac{1}{2\sqrt{u}}\left\langle S^{TX}(f_\alpha)f_\beta,g_\gamma\right\rangle c(f_\beta)g^\gamma\\
&\qquad+\frac{1}{4u}\left\langle S^{TX}(f_\alpha)g_\beta,g_\gamma\right\rangle g^\beta g^\gamma-\frac{z}{2\sqrt{u}}\hat{c}(f_\alpha)\Big)^2-\frac{u}{4}r^X\\
&-\frac{u}{8}\Big(\left\langle f_\gamma,R^{TX}(f_\alpha,f_\beta)f_\delta\right\rangle\hat{c}(f_\gamma)\hat{c}(f_\delta)-\left(\omega^{\mathscr{E}_p}\right)^2(f_\alpha,f_\beta)\Big)c(f_\alpha)c(f_\beta)\\
&-\frac{\sqrt{u}}{4}\Big(\left\langle f_\gamma,R^{TX}(f_\alpha,g_\beta)f_\delta\right\rangle\hat{c}(f_\gamma)\hat{c}(f_\delta)-\left(\omega^{\mathscr{E}_p}\right)^2(f_\alpha,g_\beta)\Big)c(f_\alpha)g^\beta\\
&-\frac{1}{8}\Big(\left\langle f_\gamma,R^{TX}(g_\alpha,g_\beta)f_\delta\right\rangle\hat{c}(f_\gamma)\hat{c}(f_\delta)-\left(\omega^{\mathscr{E}_p}\right)^2(g_\alpha,g_\beta)\Big)g^\alpha g^\beta\\
&-\frac{u}{4}\left(\omega^{\mathscr{E}_p}(f_\alpha)\right)^2-\frac{u}{8}\left(\omega^{\mathscr{E}_p}\right)^2(f_\alpha,f_\beta)\hat{c}(f_\alpha)\hat{c}(f_\beta)+\frac{u}{2}\Big[\nabla_{f_\alpha}^{\mathscr{E}_p,\mathrm{u}},\omega^{\mathscr{E}_p}(f_\beta)\Big]c(f_\alpha)\hat{c}(f_\beta)\\
&+\frac{\sqrt{u}}{2}\Big[\nabla_{g_\alpha}^{\mathscr{E}_p,\mathrm{u}},\omega^{\mathscr{E}_p}(f_\beta)\Big]g^\alpha\hat{c}(f_\beta)+\frac{\sqrt{u}z}{2}\omega^{\mathscr{E}_p}(f_\alpha)c(f_\alpha)+\frac{z}{2}\omega^{\mathscr{E}_p}(g_\alpha)g^\alpha\,.
\end{aligned}
\tag{2.5.80}
$$

By (2.5.75), (2.5.79) and (2.5.80), we get (2.5.73). □

2.5.3. *Proof of* (2.4.79).

If S is a point, (2.4.79) could be proved in the same way as [BeB94, §5]. This subsection will follow the idea of [BeB94, §5] while keeping track of the contribution of $\Lambda^\cdot(T^*S)$.

In the sequel, p is fixed and always supposed to be large enough.

In this subsection, we work with a fixed $u>0$.

We recall that P_p and $P_p^\perp$ are defined in §2.5.1. Set

$$
\begin{aligned}
B_1&=P_p\mathfrak{D}_{t,u}^{\mathscr{F}_p,[0]}P_p\,,\quad B_2=P_p\mathfrak{D}_{t,u}^{\mathscr{F}_p,[0]}P_p^\perp\,,\\
B_3&=P_p^\perp\mathfrak{D}_{t,u}^{\mathscr{F}_p,[0]}P_p\,,\quad B_4=P_p^\perp\mathfrak{D}_{t,u}^{\mathscr{F}_p,[0]}P_p^\perp\,.
\end{aligned}
\tag{2.5.81}
$$

Then

$$
\mathfrak{D}_{t,u}^{\mathscr{F}_p,[0]}=\begin{pmatrix}B_1&B_2\\B_3&B_4\end{pmatrix}.
\tag{2.5.82}
$$

By (2.4.58), we have

$$
B_1=\mathfrak{D}_u^{\mathscr{H}_p,[0]}\,,\quad -B_1^2=\mathfrak{C}_u^{\mathscr{H}_p,[0],2}\,.
\tag{2.5.83}
$$

For any operator A acting on a Hilbert space, its Schauder r-norm ($r\geqslant 1$) is defined as follows

$$
\|A\|_r=\Big(\operatorname{Tr}\left[(A^*A)^{r/2}\right]\Big)^{1/r}.
\tag{2.5.84}
$$

These norms satisfy the Hölder's inequality : for $r_1,r_2,r_3\geqslant 1$ with $1/r_1+1/r_2=1/r_3$, we have

$$
\|A\|_{r_1}\|B\|_{r_2}\geqslant\|AB\|_{r_3}\,.
\tag{2.5.85}
$$

Lemma 2.5.8. *There exist $a>0$, $b>0$ such that, the following estimates hold for $t\geqslant 1$,*

$$
-\mathfrak{D}_{t,u}^{\mathscr{F}_p,[0],2}\geqslant a^2\,,\quad -B_1^2\geqslant a^2\,,\quad \|B_2\|_\infty=\|B_3\|_\infty\leqslant b\,,\quad -B_4^2\geqslant a^2t\,.
\tag{2.5.86}
$$

Proof. The first inequality in (2.5.86) follows from (2.5.23).

The second inequality in (2.5.86) follows from (2.5.67) and (2.5.83).

Since $B_3 = -B_2^*$, we have $\|B_2\|_\infty = \|B_3\|_\infty$.

Since $D_{\mathrm{v}}^{\mathscr{F}_p} P_p = 0$, by (2.4.38), we get

$$B_3 = \sqrt{u} P_p^\perp D_{\mathrm{h}}^{\mathscr{F}_p} P_p \; . \tag{2.5.87}$$

Since $\nabla^{\mathscr{E}_p}$ preserves H_p, by (2.4.42) and (2.5.87), we get

$$B_3 = \sqrt{u} P_p^\perp \left(-\frac{1}{2}\hat{c}(f_\alpha)\omega^{\mathscr{E}_p}(f_\alpha) + \frac{1}{2}c(f_\alpha)\omega^{\mathscr{E}_p}(f_\alpha) \right) P_p \; , \tag{2.5.88}$$

which is independent of $t \geqslant 1$. This proves the third inequality in (2.5.86).

We recall that A_4 is defined by (2.5.19). By (2.4.32), we get

$$A_4 = -B_4^2 - B_3 B_2 \; . \tag{2.5.89}$$

By (2.5.21), (2.5.89) and the third inequality in (2.5.86), we get the fourth inequality in (2.5.86). □

Set

$$U = \left\{ \lambda \in \mathbb{C} \; : \; |\mathrm{Im}(\lambda)| > \frac{a}{2} \; , \; |\mathrm{Re}(\lambda)| < \frac{1}{\sqrt{3}} |\mathrm{Im}(\lambda)| \right\} . \tag{2.5.90}$$

For $\lambda \in \partial U$, put

$$\begin{aligned} E(\lambda) &= \begin{pmatrix} E_1(\lambda) & E_2(\lambda) \\ E_3(\lambda) & E_4(\lambda) \end{pmatrix} \\ &:= \begin{pmatrix} \lambda - B_1 & -B_2 \\ -B_3 & \lambda - B_4 \end{pmatrix}^{-1} - \begin{pmatrix} (\lambda - B_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p,[0]}\right)^{-1} - \left(\lambda - \mathfrak{D}_{u}^{\mathscr{H}_p,[0]}\right)^{-1} , \end{aligned} \tag{2.5.91}$$

By proceeding as in [BeB94, (5.85)], we have

$$\begin{aligned} E_1(\lambda) \; &= \left(\left(1 - (\lambda - B_1)^{-1} B_2 (\lambda - B_4)^{-1} B_3 \right)^{-1} - 1 \right) (\lambda - B_1)^{-1} \; , \\ E_2(\lambda) \; &= \left(1 - (\lambda - B_1)^{-1} B_2 (\lambda - B_4)^{-1} B_3 \right)^{-1} (\lambda - B_1)^{-1} B_2 (\lambda - B_4)^{-1} \; , \\ E_3(\lambda) \; &= (\lambda - B_4)^{-1} B_3 \left(1 - (\lambda - B_1)^{-1} B_2 (\lambda - B_4)^{-1} B_3 \right)^{-1} (\lambda - B_1)^{-1} \; , \\ E_4(\lambda) \; &= \left(1 - (\lambda - B_4)^{-1} B_3 (\lambda - B_1)^{-1} B_2 \right)^{-1} (\lambda - B_4)^{-1} \; . \end{aligned} \tag{2.5.92}$$

We fix $r \geqslant \dim Y + 1$.

Lemma 2.5.9. *There exists $C > 0$ such that, for $\lambda \in U$ and $t > 16b^4/a^4$, we have*

$$\|E(\lambda)\|_\infty \leqslant \frac{C}{\sqrt{t}} \; , \quad \|E(\lambda)\|_r \leqslant C \; . \tag{2.5.93}$$

Proof. By Lemma 2.5.8, for $\lambda \in \partial U$ and $t \geqslant 1$, we have

$$\begin{aligned}
\left\| (\lambda - B_1)^{-1} \right\|_\infty &\leqslant \frac{2}{a} \,, \\
\left\| (\lambda - B_4)^{-1} \right\|_\infty &\leqslant \frac{2}{a}\frac{1}{\sqrt{t}} \,, \\
\left\| (\lambda - B_1)^{-1} B_2 (\lambda - B_4)^{-1} B_3 \right\|_\infty &\leqslant \frac{4b^2}{a^2}\frac{1}{\sqrt{t}} \,, \\
\left\| (\lambda - B_4)^{-1} B_3 (\lambda - B_1)^{-1} B_2 \right\|_\infty &\leqslant \frac{4b^2}{a^2}\frac{1}{\sqrt{t}} \,.
\end{aligned} \tag{2.5.94}$$

By (2.5.92), (2.5.94), we get the first inequality in (2.5.93).

Since $\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}$ is a first order elliptic operator, by (2.4.32) and Corollary 2.5.5, there exists $C > 0$ such that, for $\lambda \in \partial U$ and $t \geqslant 1$, we have

$$\left\| \left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u} \right)^{-1} \right\|_r \leqslant 2 \left\| \left(\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u} \right)^{-1} \right\|_r \leqslant C \,. \tag{2.5.95}$$

By Lemma 2.5.8 and (2.5.95), there exists $C > 0$ such that, for $t \geqslant 1$, we have

$$\begin{aligned}
&\left\| \begin{pmatrix} \lambda - B_1 & 0 \\ 0 & \lambda - B_4 \end{pmatrix}^{-1} \right\|_r \\
&\leqslant \left\| \left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u} \right)^{-1} \right\|_r \left\| \begin{pmatrix} 1 & -(\lambda - B_1)^{-1} B_2 \\ -(\lambda - B_4)^{-1} B_3 & 1 \end{pmatrix} \right\|_\infty \leqslant C \,.
\end{aligned} \tag{2.5.96}$$

As a consequence,

$$\left\| (\lambda - B_1)^{-1} \right\|_r \leqslant C \,, \quad \left\| (\lambda - B_4)^{-1} \right\|_r \leqslant C \,. \tag{2.5.97}$$

By (2.5.92), (2.5.94), (2.5.97), we get the second inequality in (2.5.93). □

Let $\mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u}$ be the positive degree component of $\mathfrak{D}^{\mathscr{F}_p}_{t,u}$, i.e.,

$$\mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u} = \frac{1}{2}\omega^{\mathscr{F}_p} - \frac{1}{4\sqrt{u}}\hat{c}(T) \,. \tag{2.5.98}$$

We have

$$\begin{aligned}
\left(\lambda - \mathfrak{D}^{\mathscr{F}_p}_{t,u} \right)^{-1} &= \left(1 - \left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u} \right)^{-1} \mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u} \right)^{-1} \left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u} \right)^{-1} \\
&= \left\{ \sum_{j=0}^{\dim S} \left(\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u} \right)^{-1} \mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u} \right)^j \right\} \left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u} \right)^{-1} \,.
\end{aligned} \tag{2.5.99}$$

For $\left(\lambda - \mathfrak{D}^{\mathscr{H}_p}_u \right)^{-1}$, the same expansion holds, i.e., we replace $\mathfrak{D}^{\mathscr{F}_p}_{t,u}$, $\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}$, $\mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u}$ by $\mathfrak{D}^{\mathscr{H}_p}_u$, $\mathfrak{D}^{\mathscr{H}_p,[0]}_u$, $\mathfrak{D}^{\mathscr{H}_p,[>0]}_u$ in (2.5.99). Moreover, we have

$$\mathfrak{D}^{\mathscr{H}_p,[>0]}_u = P_p \mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u} P_p \,. \tag{2.5.100}$$

Proof of (2.4.79). Let $f(\lambda) = \lambda \exp(\lambda^2)$. Let $f_r : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be the unique holomorphic function such that

$$\frac{1}{r!}\frac{d^r}{d\lambda^r} f_r(\lambda) = f(\lambda) \,, \quad \lim_{\lambda \to +i\infty} f_r(\lambda) = \lim_{\lambda \to -i\infty} f_r(\lambda) = 0 \,. \tag{2.5.101}$$

There exists $C_r > 0$ such that for $\lambda \in U$, we have

$$\big|f_r(\lambda)\big| \leqslant C_r \exp\left(-\big|\mathrm{Im}(\lambda)\big|\right) . \tag{2.5.102}$$

We have

$$\begin{aligned} f\left(\mathfrak{D}^{\mathscr{F}_p}_{t,u}\right) &= \frac{1}{2\pi i}\int_{\partial U} f_r(\lambda)\Big(\lambda - \mathfrak{D}^{\mathscr{F}_p}_{t,u}\Big)^{-1-r} d\lambda \,, \\ f\left(\mathfrak{D}^{\mathscr{H}_p}_{u}\right) &= \frac{1}{2\pi i}\int_{\partial U} f_r(\lambda)\Big(\lambda - \mathfrak{D}^{\mathscr{H}_p}_{u}\Big)^{-1-r} d\lambda \,. \end{aligned} \tag{2.5.103}$$

Using (2.5.91), (2.5.99), (2.5.100), we can express

$$\Big(\lambda - \mathfrak{D}^{\mathscr{F}_p}_{t,u}\Big)^{-1-r} - \Big(\lambda - \mathfrak{D}^{\mathscr{H}_p}_{u}\Big)^{-1-r} \tag{2.5.104}$$

in terms of the following operators

$$\begin{aligned} & E_j(\lambda) \,, \qquad \text{for } j = 1,2,3,4 \,, \\ & \Big(\lambda - \mathfrak{D}^{\mathscr{H}_p}_{u}\Big)^{-1-r} , \qquad Q\mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u}Q' \,, \qquad \text{for } Q, Q' \in \Big\{P_p, P_p^{\perp}\Big\} \,. \end{aligned} \tag{2.5.105}$$

Moreover, the operators in the second line of (2.5.105) are independent of t. Now, applying Lemma 2.5.9 and Hölder's inequality, we can show that

$$\left\|\Big(\lambda - \mathfrak{D}^{\mathscr{F}_p}_{t,u}\Big)^{-1-r} - \Big(\lambda - \mathfrak{D}^{\mathscr{H}_p}_{u}\Big)^{-1-r}\right\|_1 \leqslant \frac{C}{\sqrt{t}} \,. \tag{2.5.106}$$

By (2.4.59), (2.4.74), (2.5.103), (2.5.106), we get the first equation in (2.4.79).

The second equation in (2.4.79) follows from the first one by the same technique as Proposition 2.3.6. □

2.5.4. *Proof of* (2.4.77).

By (2.4.32) and (2.5.23), there exists $c > 0$ such that, for $t > 0$ and $u > 0$, we have

$$\mathrm{Sp}\left(\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right) \subseteq i\Big(\,\big]-\infty, -c\sqrt{u}\big] \cup \big[c\sqrt{u}, +\infty\big[\,\Big) \,. \tag{2.5.107}$$

For $\delta > 0$, set

$$U_\delta = \Big\{\lambda \in \mathbb{C} \;:\; |\mathrm{Re}(\lambda)| < 1 \,,\; |\mathrm{Im}(\lambda)| > \delta\Big\} \,. \tag{2.5.108}$$

By (2.5.107), for $\delta < c\sqrt{u}$, we have

$$\mathrm{Sp}\left(\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right) \subseteq U_\delta \,. \tag{2.5.109}$$

We fix $r \geqslant \dim Y + 1$.

Lemma 2.5.10. *For any $\varepsilon > 0$, there exists $C > 0$, such that, for $t \geqslant 1$, $u > \varepsilon^2$ and $\lambda \notin U_{c\varepsilon/2}$, the following estimates hold*

$$\begin{aligned} \left\|\Big(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\Big)^{-1}\right\|_r &\leqslant C\big(1 + |\lambda|\big) \,, \\ \left\|\Big(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\Big)^{-1}\right\|_\infty &\leqslant C\frac{1+|\lambda|}{\sqrt{u}} \,, \\ \left\|\Big(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\Big)^{-1}\mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u}\right\|_\infty &\leqslant C \,. \end{aligned} \tag{2.5.110}$$

Proof. For $t \geqslant 1$, $u > \varepsilon^2$, $\mu \in \mathrm{Sp}\left(\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)$ and $\lambda \notin \partial U_{c\varepsilon/2}$, by (2.5.107), we have

$$\big|\lambda - \mu\big| \geqslant \min\Big\{c\varepsilon/2\,,\,1\Big\}\,. \tag{2.5.111}$$

By (2.5.111) and

$$\big|\lambda - \mu\big|^{-1} = \left|\frac{\lambda}{\lambda-\mu} - 1\right| \big|\mu\big|^{-1}\,, \tag{2.5.112}$$

there exists $C > 0$ such that

$$\big|\lambda - \mu\big|^{-1} \leqslant C\big(1 + |\lambda|\big)\big|\mu\big|^{-1}\,. \tag{2.5.113}$$

Thus

$$\begin{aligned}\left\|\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)^{-1}\right\|_r &\leqslant C\big(1+|\lambda|\big)\left\|\left(\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)^{-1}\right\|_r\,,\\ \left\|\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)^{-1}\right\|_\infty &\leqslant C\big(1+|\lambda|\big)\left\|\left(\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)^{-1}\right\|_\infty\,.\end{aligned} \tag{2.5.114}$$

By (2.5.24), we have

$$\frac{t}{2}C^{\mathscr{F}_p,2}_{\mathrm{v}} + \frac{u}{2}C^{\mathscr{F}_p,2}_{\mathrm{h}} > 0\,. \tag{2.5.115}$$

Since $\frac{t}{2}C^{\mathscr{F}_p,2}_{\mathrm{v}} + \frac{u}{2}C^{\mathscr{F}_p,2}_{\mathrm{h}}$ is a second order elliptic operator and $r > \dim Y$, we have

$$b_{t,u} := \left\|\left(\frac{t}{2}C^{\mathscr{F}_p,2}_{\mathrm{v}} + \frac{u}{2}C^{\mathscr{F}_p,2}_{\mathrm{h}}\right)^{-1/2}\right\|_r < \infty\,. \tag{2.5.116}$$

By (2.4.32) and (2.5.63), we have

$$\left\|\left(\mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)^{-1}\right\|_r \leqslant b_{t,u}\,. \tag{2.5.117}$$

By the first inequality in (2.5.114) and (2.5.117), we get

$$\left\|\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)^{-1}\right\|_r \leqslant C\big(1+|\lambda|\big)b_{t,u}\,. \tag{2.5.118}$$

Furthermore, since $C^{\mathscr{F}_p,2}_{\mathrm{v}}, C^{\mathscr{F}_p,2}_{\mathrm{h}} \geqslant 0$, $b_{t,u}$ is decreasing on t and u. This proves the first inequality in (2.5.110).

By (2.5.107) and the second inequality in (2.5.114), we get the second inequality in (2.5.110).

By (2.5.98), $\mathfrak{D}^{\mathscr{F}_p,[>0]}_{t,u}$ is a zero order differential operator whose coefficients are uniformly bounded for $u > \varepsilon^2$. Moreover, by (2.5.111),

$$\left\|\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,[0]}_{t,u}\right)^{-1}\right\|_\infty \tag{2.5.119}$$

is also uniformly bounded for $u > \varepsilon^2$. This proves the third inequality in (2.5.110). $\square$

Proof of (2.4.77). Let $f : \mathbb{C} \to \mathbb{C}$ and $f_r : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be the holomorphic functions defined by (2.5.101).

For $u > \varepsilon^2$, we have

$$f\left(\mathfrak{D}_{t,u}^{\mathscr{F}_p}\right) = \int_{\partial U_{c\varepsilon/2}} f_r(\lambda)\left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)^{-1-r} d\lambda \,. \tag{2.5.120}$$

By Lemma 2.5.10 and (2.5.99), for $\lambda \notin U_{c\varepsilon/2}$, we get

$$\begin{aligned} \left\|\left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)^{-1}\right\|_r &\leqslant C\big(1+|\lambda|\big) \,, \\ \left\|\left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)^{-1}\right\|_\infty &\leqslant C\frac{1+|\lambda|}{\sqrt{u}} \,. \end{aligned} \tag{2.5.121}$$

By Hölder's inequality and (2.5.121), we get

$$\begin{aligned} &\left|\mathrm{Tr_s}\left[\left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)^{-1-r}\right]\right| \\ &\leqslant \left\|\left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)^{-1-r}\right\|_1 \\ &\leqslant \left\|\left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)^{-1}\right\|_r^r \left\|\left(\lambda - \mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)^{-1}\right\|_\infty \leqslant C\frac{\big(1+|\lambda|\big)^{r+1}}{\sqrt{u}} \,. \end{aligned} \tag{2.5.122}$$

By (2.4.74), (2.5.120) and (2.5.122) we obtain the first equation in (2.4.77).

The second equation in (2.4.77) follows from the first one by the same transgression technique as Proposition 2.3.6. □

2.5.5. *Proof of* (2.4.78).

Following [BL95, §3], we introduce an auxiliary odd Grassmannian variable z such that $z^2 = 0$. For

$$A \in \mathrm{End}\,(\mathscr{F}_p) \otimes \Lambda^{\cdot}(T^*S) \otimes \mathbb{C}[z] \,, \tag{2.5.123}$$

we have

$$A = A_0 + zA_1 \,, \quad \text{with } A_0, A_1 \in \mathrm{End}\,(\mathscr{F}_p) \otimes \Lambda^{\cdot}(T^*S) \,. \tag{2.5.124}$$

Put

$$\mathrm{Tr}_\mathrm{s}^z\big[A\big] = \mathrm{Tr_s}\big[A_1\big] \in \Lambda^{\cdot}(T^*S) \,. \tag{2.5.125}$$

The following identity holds

$$\mathrm{Tr_s}\left[\mathfrak{D}_{t,u}^{\mathscr{F}_p} \exp\left(\mathfrak{D}_{t,u}^{\mathscr{F}_p,2}\right)\right] = \mathrm{Tr}_\mathrm{s}^z\left[\exp\left(\mathfrak{D}_{t,u}^{\mathscr{F}_p,2} + z\mathfrak{D}_{t,u}^{\mathscr{F}_p}\right)\right] \tag{2.5.126}$$

The proof of (2.4.78) is closely related to the proof of corresponding results in [BeB94, Theorem 4.13], [Ma99, Theorem 4.9], and [Ma02, Theorem 4.6].

Let $a > 0$ be the infimum of the injectivity radius of the fibers X. Let $\alpha \in \,]0, a/4]$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(x) = 1 \quad \text{for } |x| \leqslant \alpha/2 \,, \qquad \rho(x) = 0 \quad \text{for } |x| \geqslant \alpha \,. \tag{2.5.127}$$

For $\varsigma > 0$ and $z \in \mathbb{C}$, set

$$\begin{aligned} F_\varsigma(z) &= \int_{-\infty}^{+\infty} \exp\left(\sqrt{2}xz\right) \exp\left(-\frac{x^2}{2}\right) \rho(\sqrt{2}\varsigma x) \frac{dx}{\sqrt{2\pi}} \,, \\ G_\varsigma(z) &= \int_{-\infty}^{+\infty} \exp\left(\sqrt{2}xz\right) \exp\left(-\frac{x^2}{2}\right) \big(1 - \rho(\sqrt{2}\varsigma x)\big) \frac{dx}{\sqrt{2\pi}} \,. \end{aligned} \tag{2.5.128}$$

Then

$$F_\varsigma(z) + G_\varsigma(z) = \exp\left(z^2\right) \ . \tag{2.5.129}$$

Moreover, $F_\varsigma(z)$ and $G_\varsigma(z)$ take real values on $i\mathbb{R}$. As functions of $z \in i\mathbb{R}$, they lie in the Schwartz space $\mathcal{S}(i\mathbb{R})$.

The functions $F_\varsigma(z), G_\varsigma(z)$ are even holomorphic functions. Therefore there exist holomorphic functions $\widetilde{F}_\varsigma(z), \widetilde{G}_\varsigma(z)$ such that

$$F_\varsigma(z) = \widetilde{F}_\varsigma(z^2) \ , \quad G_\varsigma(z) = \widetilde{G}_\varsigma(z^2) \ . \tag{2.5.130}$$

By (2.5.129), (2.5.130), we deduce that

$$\widetilde{F}_\varsigma(z) + \widetilde{G}_\varsigma(z) = \exp(z) \ . \tag{2.5.131}$$

Put

$$\mathfrak{L}^{\mathscr{F}_p}_{t,u} = \mathfrak{D}^{\mathscr{F}_p,2}_{t,u} + z\mathfrak{D}^{\mathscr{F}_p}_{t,u}. \tag{2.5.132}$$

By (2.5.131), we get

$$\widetilde{F}_\varsigma\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right) + \widetilde{G}_\varsigma\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right) = \exp\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right) \ . \tag{2.5.133}$$

By (2.5.126), (2.5.132), (2.5.133), we obtain

$$\mathrm{Tr_s}\left[\mathfrak{D}^{\mathscr{F}_p}_{t,u}\exp\left(\mathfrak{D}^{\mathscr{F}_p,2}_{t,u}\right)\right] = \mathrm{Tr}^z_{\mathrm{s}}\left[\widetilde{F}_\varsigma\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)\right] + \mathrm{Tr}^z_{\mathrm{s}}\left[\widetilde{G}_\varsigma\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)\right] \ . \tag{2.5.134}$$

Proposition 2.5.11. *There exist $c, C > 0$ such that for $t \geqslant 1$, $0 < u \leqslant 1$, we have*

$$\left|\mathrm{Tr}^z_{\mathrm{s}}\left[\widetilde{G}_u\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)\right]\right| \leqslant C\exp\left(-c/u\right) \ . \tag{2.5.135}$$

Proof. Due to the relation

$$\frac{\partial^m}{\partial x^m}\exp\left(\sqrt{2}xz\right) = 2^{m/2}z^m\exp\left(\sqrt{2}xz\right) \ , \tag{2.5.136}$$

we can integrate by parts in the expression of $z^m G_\varsigma(z)$ and obtain that for $m \in \mathbb{N}$, there exists $C_m > 0$ such that, for $z \in \mathbb{C}$ with $|\mathrm{Re}(z)| \leqslant 1$, we have

$$\big|z\big|^m\big|G_\varsigma(z)\big| \leqslant C_m\exp\left(-\frac{\alpha^2}{8\varsigma}\right) \ . \tag{2.5.137}$$

Set

$$U = \Big\{z \in \mathbb{C} \ : \ 4\mathrm{Re}(z) + |\mathrm{Im}(z)|^2 < 4\Big\} \ . \tag{2.5.138}$$

We have

$$\sqrt{U} := \Big\{z \in \mathbb{C} \ : \ z^2 \in U\Big\} = \Big\{z \in \mathbb{C} \ : \ |\mathrm{Re}(z)| \leqslant 1\Big\} \ . \tag{2.5.139}$$

By (2.5.130), (2.5.137), (2.5.139), for $z \in U$, we have

$$\big|z\big|^{m/2}\big|\widetilde{G}_\varsigma(z)\big| \leqslant C_m\exp\left(-\frac{\alpha^2}{8\varsigma}\right) \ . \tag{2.5.140}$$

For $r \in \mathbb{N}$, let $\widetilde{G}_{r,\varsigma}(z)$ be the unique holomorphic function satisfying

$$\frac{1}{r!}\frac{d^r}{dz^r}\widetilde{G}_{r,\varsigma}(z) = \widetilde{G}_\varsigma(z) \ , \quad \lim_{z\to-\infty}\widetilde{G}_{r,\varsigma}(z) = 0 \ . \tag{2.5.141}$$

By (2.5.140), (2.5.141), for $m > 2r$, there exists $C_{m,r} > 0$ such that, for $z \in U$, we have

$$\big|\widetilde{G}_{r,\varsigma}(z)\big| \leqslant C_{m,r}\big|z\big|^{r-m/2}\exp\left(-\frac{\alpha^2}{8\varsigma}\right) . \tag{2.5.142}$$

We fix $r \geqslant (\dim Y + 1)/2$.
We have

$$\widetilde{G}_u\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right) = \frac{1}{2\pi i}\int_{\partial U}\widetilde{G}_{r,u}(\lambda)\left(\lambda - \mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)^{-r-1}d\lambda . \tag{2.5.143}$$

By (2.5.132), we have

(2.5.144)

$$\begin{aligned}\left(\lambda - \mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)^{-1} &= \left(\lambda - \mathfrak{D}^{\mathscr{F}_p,2}_{t,u}\right)^{-1} + z\mathfrak{D}^{\mathscr{F}_p}_{t,u}\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,2}_{t,u}\right)^{-2}\\ &= \left(\sqrt{\lambda} - \mathfrak{D}^{\mathscr{F}_p}_{t,u}\right)^{-1}\left(\sqrt{\lambda} + \mathfrak{D}^{\mathscr{F}_p}_{t,u}\right)^{-1}\left(1 + z\mathfrak{D}^{\mathscr{F}_p}_{t,u}\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,2}_{t,u}\right)^{-1}\right) .\end{aligned}$$

For $\mu \in \mathbb{R}$ and $\lambda \in \partial U$, we have

$$\left|i\mu\big(\lambda + \mu^2\big)^{-1}\right| \leqslant 1 . \tag{2.5.145}$$

Thus

$$\left\|z\mathfrak{D}^{\mathscr{F}_p}_{t,u}\left(\lambda - \mathfrak{D}^{\mathscr{F}_p,2}_{t,u}\right)^{-1}\right\|_{\infty} \leqslant 1 . \tag{2.5.146}$$

By (2.5.139), for $\lambda \in \partial U$, we have $\mathrm{Re}(\sqrt{\lambda}) = \pm 1$. Then the same argument for (2.5.121) show that there exists $C > 0$ such that, for $t \geqslant 1$, $0 < u \leqslant 1$ and $\lambda \in \partial U$, we have

$$\begin{aligned}\left\|\left(\sqrt{\lambda} \pm \mathfrak{D}^{\mathscr{F}_p}_{t,u}\right)^{-1}\right\|_{2r} &\leqslant C\frac{1+\sqrt{|\lambda|}}{u} ,\\ \left\|\left(\sqrt{\lambda} \pm \mathfrak{D}^{\mathscr{F}_p}_{t,u}\right)^{-1}\right\|_{\infty} &\leqslant C .\end{aligned} \tag{2.5.147}$$

Using (2.5.142), (2.5.144), (2.5.146), (2.5.147) to (2.5.143) in the same way as in the proof of (2.4.77), we complete the proof. □

Let $d_{t,u}(\cdot,\cdot)$ be the distance along the fiber Y associated with the metric $2g^{T_{\mathbb{R}}N}/t \oplus g^{TX}/u$. Using finite propagation speed of solutions of hyperbolic equations (cf. [MaMar07, §D.2]), we get

$$\widetilde{F}_u\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)(y,y') = 0 , \quad \text{for} \;\; d_{t,u}\big(y,y'\big) \geqslant \alpha/\sqrt{u} . \tag{2.5.148}$$

Let $d_X(\cdot,\cdot)$ be the distance on X associated with the metric g^{TX}. Since

$$d_{t,u}\big(y,y'\big) \geqslant u^{-1/2}d_X\big(q(y),q(y')\big) , \tag{2.5.149}$$

by (2.5.148), we get

$$\widetilde{F}_u\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)(y,y') = 0 , \quad \text{for} \;\; d_X\big(q(y),q(y')\big) \geqslant \alpha . \tag{2.5.150}$$

We will establish the following result, which combined with (2.5.134), (2.5.135) gives the first equation in (2.4.78).

Theorem 2.5.12. *There exists $\delta \in \left]0, \frac{1}{2}\right]$ such that for $t \geqslant 1, u \in]0,1]$,*

$$\sqrt{2\pi i}\varphi \operatorname{Tr}_{\mathrm{s}}^{z}\left[\widetilde{F}_u\left(\mathfrak{L}_{t,u}^{\mathscr{F}_p}\right)\right] = \pi_*\left[e\left(TX, \nabla^{TX}\right)\alpha_{p,t}\right] + \mathscr{O}\big(u^{\delta}\big). \tag{2.5.151}$$

Also the convergence in (2.5.151) *is uniform for $t \geqslant 1$.*

Proof. By (2.5.150), the proof of our theorem is local on the base X. We will proceed as in [BeB94, §7] and in [Ma99, Theorem 4.9]. More precisely, we may as well replace the base X by $T_{x_0}X$.

Given $x_0 \in X$, the exponential $\exp_{x_0}$ identifies $B_{T_{x_0}X}(0,\alpha)$ and $B_X(x_0,\alpha)$.

Also, we will use the techniques of the local families index theorem of [B86], [Ma99, §7], [Ma02, §7] to study the asymptotics of the operator $\mathfrak{L}_{u,T}^{\mathscr{F}_p}$ in the above trivialization. First we make the change of variables on $T_{x_0}X$ given by $Y \to \sqrt{u}Y$. We introduce the connection $\nabla^{\Lambda^{\cdot}(TX),u}$ along the fibres X

$$\begin{aligned}\nabla_{\cdot}^{\Lambda^{\cdot}(TX),u} = \nabla_{\cdot}^{\Lambda^{\cdot}(TX)} &+ \frac{1}{2\sqrt{u}}\left\langle S^{TX}(\cdot)f_\beta, g_\gamma\right\rangle c(f_\beta)g^\gamma \\ &+ \frac{1}{4u}\left\langle S^{TX}(\cdot)g_\beta, g_\gamma\right\rangle g^\beta g^\gamma - \frac{z}{2\sqrt{u}}\hat{c}(\cdot)\,.\end{aligned} \tag{2.5.152}$$

We trivialize the vector bundle $\mathscr{E}_p \otimes \Lambda^{\cdot}(T^*S)\,\widehat{\otimes}\Lambda^{\cdot}(T^*X)\,\widehat{\otimes}\mathbb{C}[z]$ along the geodesic $s \to \exp_{x_0}(sY)$ using the connections $\nabla^{\mathscr{E}_p,\mathrm{u}}$ and $\nabla^{\Lambda^{\cdot}(TX),u}$.

Our operator $\mathfrak{L}_{t,u}^{\mathscr{F}_p}$ will now be viewed as acting on

$$\mathscr{C}^\infty\big(B^{T_{x_0}X}, \mathscr{E}_p \otimes \Lambda^{\cdot}(T^*S)\,\widehat{\otimes}\Lambda^{\cdot}(T^*X)\,\widehat{\otimes}\mathbb{C}[z]\big)\,. \tag{2.5.153}$$

Finally, we make the Getzler rescaling, which consists in replacing the Clifford variables $c(f), f \in TX$ by $f^*/\sqrt{u} - \sqrt{u}i_f$. As usual, the operators $f^*\wedge, i_f$ now act on a different copy of the exterior algebra $\Lambda^{\cdot}(T^*X)$. We denote by $\mathfrak{L}_{t,u,x_0}^{\mathscr{F}_p}$ the operator $\mathfrak{L}_{t,u}^{\mathscr{F}_p}$ in the above trivialization.

Given $t > 0$, set

$$\begin{aligned}\mathfrak{L}_{t,0,x_0}^{\mathscr{F}_p} = \Big(\sqrt{t}D_{\mathrm{v},x_0}^{\mathscr{F}_p} + \frac{1}{2}\omega_{x_0}^{\mathscr{E}_p}\Big)^2 + z\Big(\sqrt{t}D_{\mathrm{v},x_0}^{\mathscr{F}_p} + \frac{1}{2}\omega_{x_0}^{\mathscr{E}_p}\Big) \\ + \Big(\partial_\alpha + \frac{1}{4}\left\langle R_{x_0}^{TX}Y, f_\alpha\right\rangle\Big)^2 - \frac{1}{4}\left\langle f_\gamma, R_{x_0}^{TX}f_\delta\right\rangle \hat{c}(f_\gamma)\hat{c}(f_\delta)\,.\end{aligned} \tag{2.5.154}$$

Using the same arguments as in [Ma99, (7.23)], [Ma02, (7.21)], from (2.5.73), we deduce that as $u \to 0$,

$$\mathfrak{L}_{t,u,x_0}^{\mathscr{F}_p} = \mathfrak{L}_{t,0,x_0}^{\mathscr{F}_p} + \mathscr{O}\big(\sqrt{u}\big)\,. \tag{2.5.155}$$

The convergence above is a uniform convergence over compact sets of the coefficients of the considered differential operators on compact subsets together with their derivatives of arbitrary order. Note that the coefficients of the operator $\mathscr{O}(\sqrt{u})$ are in general unbounded.

To establish (2.5.151), we will briefly show how to replace the fibration $\pi : M \to S$ by a fibration by vector spaces. Let U be a small open set in S and let $s_0 \in U$ be such that $\pi^{-1}U \simeq U \times X_{s_0}$. Let $x_0 \in X_{s_0}$. Then $U \times \{x_0\}$ is a section of M over U. Using geodesic coordinates along the fibers X based at the section x_0, we have identified a neighborhood V of $U \times \{x_0\}$ in M with a neighbourhood of the zero section in $U \times T_{x_0}X$. Let $g^{T_{x_0}X}$ be the given metric on $T_{x_0}X$. The fibres of V are now equipped with two distinct metrics: one induced by the given metric g^{TX}, and the other by the constant metric $g^{T_{x_0}X}$. Set

$$\widetilde{g}^{TX} = \rho(|Y|/2)\, g^{TX} + (1-\rho(|Y|/2))\, g^{T_{x_0}X}. \tag{2.5.156}$$

In the same way, we can extend $T^H M$ on a neighbourhood of the zero section of $S \times T_{x_0}X$ to a full horizontal vector bundle on $S \times T_{x_0}X$ which will just TS for $|Y| \geqslant \alpha$. Similarly, the flat fibration $q : \mathcal{N} \to M$ induces a corresponding flat fibration over $U \times T_{x_0}X$.

This way, we can construct an operator $\overline{\mathfrak{L}}^{\mathscr{F}_p}_{t,u}$ over $U \times T_{x_0}X$ which coincides with $\mathfrak{L}^{\mathscr{F}_p}_{t,u}$ for $|Y| \leqslant \alpha$. Because of this, if $y \in \mathcal{N}, q(y) = x_0$, we have the identity

$$\widetilde{F}_u\left(\mathfrak{L}^{\mathscr{F}_p}_{t,u}\right)(y,y) = \widetilde{F}_u\left(\overline{\mathfrak{L}}^{\mathscr{F}_p}_{t,u}\right)(y,y). \tag{2.5.157}$$

The advantage of dealing with $\overline{\mathfrak{L}}^{\mathscr{F}_p}_{t,u}$ is that the dilation $Y \to \sqrt{u}Y$ can now be made on the full vector space $T_{x_0}X$.

We can now proceed exactly as in [BL91, §13] and in [BeB94, §9 d)–9 g)] to establish (2.5.151) at least when $t \geqslant 1$ remains bounded.

We will now show how to obtain uniformity in (2.5.151) for $t \geqslant 1$. We will follow closely the arguments in [BeB94, §9(b), 9(c)], which are inspired from [BL91, §13]. Recall that P_p denotes the orthogonal projection from $\mathscr{E}_p$ on H_p. We still denote by P_p the corresponding projection from $\mathscr{F}_p$ on $\mathscr{H}_p$. As in §2.5.1, we will write the operator $\mathfrak{L}^{\mathscr{F}_p}_{t,u}$ as a $(2,2)$ matrix with respect to the splitting $\mathscr{F}_p = \mathscr{H}_p \oplus \mathscr{H}_p^{\perp}$. With respect to this splitting, given $u > 0$, as $t \to +\infty$, $\mathfrak{L}^{\mathscr{F}_p}_{t,u}$ as the preferred matrix structure

$$\mathfrak{L}^{\mathscr{F}_p}_{t,u} = \begin{bmatrix} \mathscr{O}(1) & \mathscr{O}\left(\sqrt{t}\right) \\ \mathscr{O}\left(\sqrt{t}\right) & \mathscr{O}(t) \end{bmatrix}. \tag{2.5.158}$$

Given $u > 0$, we can proceed exactly as in [BeB94, §9] to give another proof of Theorem 2.4.9.

We will now show how to use the above techniques to obtain the required uniformity in (2.5.151). The difficulty is to combine the local index theoretic techniques over X that were described above with the splitting $\mathcal{E}_p = H_p \oplus H_p^{\perp}$. Let $\nabla^{\mathscr{E}_p,u,s}$ denote the connection on $\mathscr{E}_p$ which is the orthogonal projection of $\nabla^{\mathscr{E}_p,u}$ with respect to the splitting $\mathscr{E}_p = H_p \oplus H_p^{\perp}$. In the constructions that were given at the beginning of our proof, we will instead trivialize $\mathscr{E}_p$ by parallel transport with respect to the connection $\nabla^{\mathscr{E}_p,u,s}$. We make exactly the same Getzler rescalings as before, while keeping track of the splitting of $\mathscr{E}_{x_0}$. The situation is indeed exactly the same as in [BeB94, §9]. Using the above splitting ultimately guarantees that the resolvent of $\mathfrak{L}^{\mathscr{F}_p}_{t,u,x_0}$ can be uniformly controlled for $t \geqslant 1$ as $u \to 0$.

This completes the proof of our theorem. □

Proof of (2.4.78). The first equation in (2.4.78) follows from (2.5.134), (2.5.135), (2.5.151).

The second equation in (2.4.78) follows from the first one by the same transgression technique as Proposition 2.3.6. □

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Index

École doctorale de mathématiques Hadamard (EDMH)

Titre: Limites adiabatiques, fibrations holomorphes plates et théorème de R.R.G.

Mots clés: torsion analytique, théorie de la diffusion, théorème de l'indice, caractéristique de Chern

Résume: Cette thèse est faite de deux parties. La première partie est un article rédigé conjointement avec Martin Puchol et Jialin Zhu. La deuxième partie est une série de résultats obtenus par moi-même liés au théorème de Riemann-Roch-Grothendieck pour les fibrés vectoriels plats.

Dans la première partie, nous donnons une preuve analytique d'un résultat décrivant le comportement de la torsion analytique en théorie de de Rham lorsque la variété considérée est séparée en deux par une hypersurface. Plus précisément, nous donnons une formule liant la torsion analytique de la variété entière aux torsions analytiques associées aux variétés à bord avec des conditions limites relative ou absolue le long de l'hypersurface.

Dans la deuxième partie de cette thèse, nous raffinons les résultats de Bismut-Lott pour les images directes des fibrés vectoriels plats au cas où le fibré vectoriel plat en question est lui-même la cohomologie holomorphe d'un fibré vectoriel le long d'une fibration plate à fibres complexes. Dans ce contexte, nous donnons une formule de Riemann-Roch-Grothendieck dans laquelle la classe de Todd du fibré tangent relatif apparaît explicitement. En remplaçant les classes de cohomologie par des formes explicites qui les représentent en théorie de Chern-Weil, nous généralisons ainsi des constructions de Bismut-Lott.

Title: Adiabatic limits, flat holomorphic fibrations and R.R.G. Theorem

Keywords: analytic torsion, scattering theory, index theorem, Chern characters

Abstract: This thesis consists of two parts. The first part is an article written jointly with Martin Puchol and Jialin Zhu, the second part is a series of results obtained by myself in connection with the Riemann-Roch-Grothendieck theorem for flat vector bundles.

In the first part, we give an analytic approach to the behavior of classical Ray-Singer analytic torsion in de Rham theory when a manifold is separated along a hypersurface. More precisely, we give a formula relating the analytic torsion of the full manifold, and the analytic torsion associated with relative or absolute boundary conditions along the hypersurface.

In the second part of this thesis, we refine the results of Bismut-Lott on direct images of flat vector bundles to the case where the considered flat vector bundle is itself the fiberwise holomorphic cohomology of a vector bundle along a flat fibration by complex manifolds. In this context, we give a formula of Riemann-Roch-Grothendieck in which the Todd class of the relative holomorphic tangent bundle appears explicitly. By replacing cohomology classes by explicit differential forms in Chern-Weil theory, we extend the constructions of Bismut-Lott in this context.